

EXISTENCE AND NON-EXISTENCE RESULTS FOR A LOGISTIC-TYPE EQUATION ON MANIFOLDS

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Dedicated to the memory of Franca Burrone Rigoli

ABSTRACT. We study the steady state solutions of a generalized logistic type equation on a complete Riemannian manifold. We provide sufficient conditions for existence, respectively non-existence of positive solutions, which depend on the relative size of the coefficients and their mutual interaction with the geometry of the manifold, which is mostly taken into account by means of conditions on the volume growth of geodesic balls.

0. INTRODUCTION

The aim of this paper is to study the problem of existence, non-existence and uniqueness of solutions of the equation

$$(0.1) \quad \Delta u + a(x)u - b(x)u^\sigma = 0 \quad \text{on } M,$$

on a complete, connected Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Here $\sigma > 1$, the coefficient $b(x)$ is assumed to be non-negative while $a(x)$ is not assumed to be of constant sign.

Equations of the form (0.1) arise in Riemannian geometry, as the equation for the change of the scalar curvature under a conformal change of the metric (see, e.g., [K]) and in mathematical biology, where they describe the steady state solutions of the logistic equation with diffusion

$$(0.2) \quad \frac{\partial}{\partial t} u = \Delta u + a(x)u - b(x)u^\sigma,$$

(see, e.g., [AB], [AW], [DM1], [DM2]). In the latter context u represents the density of a population, and it is therefore assumed to be nonnegative, the non-linear term $-b(x)u^\sigma$ accounts for the fact that the population is self limiting, and the function $a(x)$ represents the birth rate of the population, with no self limitation.

In Euclidean setting, G.A. Afrouzi and K.J Brown, [AB], have studied the following special case of (0.1)

$$(0.3) \quad \Delta u + \lambda[g(x)u - u^2] = 0 \quad \text{on } \mathbb{R}^m,$$

where the positive parameter $\lambda > 0$ is the inverse of the diffusion rate, and $g(x)$ is a changing sign coefficient which again represents the birth rate of the population. Their results describe the interplay between diffusion, and birth rate, and show that if diffusion is sufficiently small, solutions may exist even if a is predominantly

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negative, while if diffusion is large, then solutions exist only if the birth rate is sufficiently large.

In fact, the mutual interactions between diffusion and birth rate is often taken into account by the principal eigenvalue λ_* of the linear part of equation (0.3) (see Section 1 below for the relevant definitions). This is exemplified, e.g., in [AB] Theorem 2.2, where it is proved that if g is positive somewhere, so that $\lambda_* \geq 0$, and if $\lambda > \lambda_*$, then equation (0.3) has a positive solution.

Further, (see Section 3 therein) under the additional assumption that $g(x)$ is strictly negative in the complement of a ball, (0.3) has (exactly one) positive solution if $\lambda > \lambda_*$, and no positive solution if $\lambda \leq \lambda_*$.

More recently, existence, non-existence, and uniqueness results have been obtained by Y. Du and L. Ma, [DM2], who study the equation

$$(0.4) \quad \Delta u + \lambda g(x)u - b(x)u^\sigma = 0 \quad \text{on } \mathbb{R}^m$$

where $\sigma > 1$ and $b(x)$ is a non-negative coefficient. They show that if g is positive somewhere and its positive part satisfies suitable conditions which in particular imply that $\lambda_* > 0$, and if b is strictly positive outside a bounded connected open set S_o with principal eigenvalue $\lambda_1(S_o)$ ($= +\infty$ if S_o is empty), then (0.4) has a (unique) solution in the homogeneous Sobolev space $H_h^1(\mathbb{R}^m)$ if $\lambda_* < \lambda < \lambda_1(S_o)$, no solution in $H_h^1(\mathbb{R}^m)$ if $0 < \lambda \leq \lambda_*$ and no solutions at all if $\lambda_1(S_o) < +\infty$ and $\lambda \geq \lambda_1(S_o)$.

Our results provide new insight on the interplay between diffusion and growth rate, that is, in our notation, between the relative size of the variable coefficients $a(x)$ and $b(x)$. Indeed, we show that if $a(x)$ is sufficiently large in a suitable ball, while outside the ball the negative part of $a(x)$ is not too big (so a possibly overall negative birth rate is compensated by a sufficiently large positive birth rate in the ball), then (0.1) has a positive solution (see Theorem 2.10 below), independently of the size of $b(x)$. On the other hand, the content of Theorem 3.7 is that if $a(x)$ is sufficiently small compared to $b(x)$, and certain geometric conditions on the volume growth of the manifold hold, then (0.1) has no positive solution. We note that in the special case where b is constant, and the underlying manifold is Euclidean space, Theorem 3.7 generalizes and complements the non-existence result contained in [AB], Section 3.

Observe also that, according to Theorem 2.1 in [BRS2], if $a(x)$ is sufficiently large that the bottom of the spectrum $\lambda_1^{\Delta+a(x)}(M)$ of the Schrödinger operator $\Delta + a(x)$ is negative, then one can guarantee the existence of a (minimal) solution of (0.1) irrespectively of the size of $b(x)$. This is tightly related to the above mentioned relationship between the existence of steady state solutions and the principal eigenvalue λ_* of the problem $\Delta u + \lambda a(x)u = 0$ on M . Indeed, as we shall explain in Section 1, λ_* is precisely the largest value of λ for which $\lambda_1^{\Delta+\lambda a(x)}(M) \geq 0$.

Note however, that in our main existence result, Theorem 2.10, we avoid an assumption of this type and describe explicit conditions on the coefficients that guarantee existence, thus giving a new contribution to the subject.

It should also be stressed that having replaced Euclidean space with a Riemannian manifold, the behavior of the equation is now sensitive to the geometry of the underlying space, and therefore reflects not only the mutual relationship of the coefficients $a(x)$ and $b(x)$, but also their respective interaction with the geometry.

From the analytic point of view this introduces new difficulties. For instance, to prove our main non-existence result we need to determine an asymptotic a priori upper bound for the solution u , and the techniques that are usually employed in Euclidean setting are not available. We overcome the problem via a different approach of geometric flavor, which may be of independent interest (see Lemma 3.6).

From now on we denote by $(M, \langle \cdot, \cdot \rangle)$ a connected, complete, non-compact Riemannian manifold of dimension $m \geq 2$. We fix a reference point o in M , and denote by $r(x)$ the Riemannian distance function from o , and by B_R and ∂B_R the geodesic ball and sphere, respectively, of radius $R > 0$ centered at o . Finally, we will denote by C , possibly with subscripts or superscripts, a positive constant which may vary from place to place and that may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. Given functions A and B , defined on a set Ω , we say that $A = O(B)$ in Ω if there exists C such that

$$A(t) \leq CB(t) \quad \forall t \in \Omega.$$

1. ON THE PRINCIPAL EIGENVALUE λ_*

As mentioned in the Introduction, existence results for equation (0.3) typically depend on the assumption that the parameter λ be strictly greater than the principal eigenvalue of the linear part of the equation.

Recall that a constant λ_1 is said to be a principal eigenvalue for the linear equation

$$\Delta u + \lambda a(x)u = 0$$

if for $\lambda = \lambda_1$ the equation has a positive solution.

On the other hand, in the literature on the non-compact Yamabe equation existence results often depend on the assumption that the sign of the bottom of the spectrum of the Schrödinger operator associated to the equation be negative (see, e.g. [BRS2]).

It is therefore natural to investigate the relationships between principal and spectral eigenvalues.

Let $a(x) \in C^\infty(M)$ and, given a fixed radius R , consider the eigenvalue problem

$$(1.1) \quad \begin{cases} \Delta \varphi + \lambda a(x)\varphi = 0 & \text{on } B_R, \lambda \in \mathbb{R} \\ \varphi = 0 & \text{on } \partial B_R. \end{cases}$$

If $a(x_o) > 0$ for some $x_o \in B_R$, then it is well known (see [MM], [HK]) that (1.1) has a positive principal eigenvalue $\lambda_1(B_R)$, which is variationally characterized by

$$(1.2) \quad \lambda_1(R) = \inf \left\{ \int_{B_R} |\nabla u|^2 : u \in H_o^1(B_R), \int_{B_R} a(x)u^2 = 1 \right\},$$

and a principal positive eigenfunction φ on B_R satisfying

$$(1.3) \quad \begin{cases} \Delta \varphi + \lambda_1(R)a(x)\varphi = 0 & \text{on } B_R, \\ \varphi = 0 & \text{on } \partial B_R. \end{cases}$$

We note in passing that, by the maximum principle, the condition that $a(x)$ is positive somewhere in B_R is also necessary for the existence of a positive principal eigenvalue.

It follows from (1.2) that $\lambda_1(R)$ is a non-increasing function of R , and we may set

$$(1.4) \quad \lambda_* = \lim_{R \rightarrow +\infty} \lambda_1(R) \geq 0.$$

On the other hand, for $\mu \in \mathbb{R}$, let L_μ be the operator $L_\mu = \Delta + \mu a(x)$ and denote by $\lambda_1(L_\mu, R)$ the first Dirichlet eigenvalue of L_μ on B_R , so that

$$\lambda_1(L_\mu, R) = \inf \left\{ \int_{B_R} |\nabla u|^2 - \mu a(x) u^2 : u \in H_o^1(B_R), \int_{B_R} u^2 = 1 \right\},$$

and there exists a smooth positive eigenfunction ψ of L_μ on B_R satisfying

$$(1.5) \quad \begin{cases} L_\mu \psi = -\lambda_1(L_\mu, R) \psi & \text{on } B_R, \\ \psi = 0 & \text{on } \partial B_R. \end{cases}$$

Again $\lambda_1(L_\mu, R)$ is a non-increasing function of R and one may define

$$\lambda_1^{L_\mu}(M) = \lim_{R \rightarrow +\infty} \lambda_1(L_\mu, R),$$

which coincides with the bottom of the L^2 -spectrum of L_μ in the case where the operator is essentially self-adjoint on $C_c^\infty(M)$ (this happens, e.g., if the operator L_μ is bounded from below on C_c^∞ , see [BdCS], Proposition 2).

By a result of W.F. Moss and J. Pieperbrink, [MP], and D. Fisher-Colbrie and R. Schoen, [FCS], we have that $\lambda_1^{L_\mu}(M) \geq 0$, if and only if there exists a positive solution $u \in C^\infty(M)$ of

$$(1.6) \quad \Delta u + \mu a(x) u = 0$$

on M .

We are now ready to prove the following

Proposition 1.1. *Let $a(x) \in C^\infty(M)$ satisfy $a(x_o) > 0$ for some $x_o \in M$. Then*

$$\lambda_* = \sup\{\mu \geq 0 : \lambda_1^{L_\mu}(M) \geq 0\}.$$

Proof. Fix R_o sufficiently large that $x_o \in B_{R_o}$, and choose a sequence R_k such that $R_o < R_k \nearrow +\infty$. Denote by φ_k the solution of (1.3) on B_{R_k} with principal eigenvalue $\lambda_1(R_k)$, normalized with $\varphi_k(x_o) = 1$. Arguing as in the proof of [J], Theorem 1, one shows that $\{\varphi_k\}$ has a subsequence which converges locally uniformly on M to a C^∞ non-negative function φ satisfying $\varphi(x_o) = 1$ and

$$\Delta \varphi + \lambda_* a(x) \varphi = 0 \quad \text{on } M.$$

Furthermore, by the maximum principle (see [GT], p. 35), $\varphi > 0$ on M . It follows from (1.6) that $\lambda_1^{L_{\lambda_*}}(M) \geq 0$, so that

$$\lambda_* \leq \sup\{\mu \geq 0 : \lambda_1^{L_\mu}(M) \geq 0\}.$$

On the other hand, let $\mu \geq 0$ be such that $\lambda_1^{L_\mu}(M) \geq 0$. We claim that $\mu \leq \lambda_*$ so that the reverse inequality holds in the above formula, and the required conclusion follows. To this end, let u be a smooth positive function satisfying (1.6), and fix $R > 0$ sufficiently large that $x_o \in B_R$. Defining $w = \log u$, it follows from (1.6) that

$$(1.7) \quad \Delta w = -\mu a(x) - |\nabla w|^2.$$

Given any $v \in C_0^\infty(B_R)$, $v \not\equiv 0$, we multiply both sides of (1.7) times v^2 , integrate by parts and use Young inequality to obtain

$$\int_{B_R} \mu a(x) v^2 + |\nabla w|^2 v^2 = \int_{B_R} 2v \langle \nabla v, \nabla w \rangle \leq \int_{B_R} |\nabla w|^2 v^2 + |\nabla v|^2,$$

whence

$$\mu \int_{B_R} a(x) v^2 \leq \int_{B_R} |\nabla v|^2.$$

Now the variational characterization of the principal eigenvalue shows that $\mu \leq \lambda_1(R)$ and the claim follows from the definition of λ_* . \square

Corollary 1.2. *Let $a(x) \in C^\infty(M)$ be such that $a(x_o) > 0$ for some $x_o \in M$. Then a non-negative number μ satisfies $\mu > \lambda_*$ if and only if $\lambda_1^{L_\mu}(M) < 0$.*

Proof. Since in our assumptions $\lambda_* \geq 0$, we may assume that $\mu > 0$. Assume by contradiction that $\lambda_1^{L_\mu}(M) < 0$ and $\mu \leq \lambda_*$. By definition, there exists R sufficiently large that $x_o \in B_R$ and $\lambda_1(L_\mu, R) < 0$, so that, if ψ is the corresponding positive eigenfunction as in (1.5), we have

$$\int_{B_R} |\nabla \psi|^2 \leq \mu \int_{B_R} a(x) \psi^2.$$

In particular, the integral on the right hand side is positive, and since $\psi \in H_0^1(B_R)$, we have

$$\lambda_* \leq \lambda_1(R) \leq \frac{\int_{B_R} |\nabla \psi|^2}{\int_{B_R} a(x) \psi^2} < \mu,$$

which gives the required contradiction. The reverse implication is an immediate consequence of Proposition 1.1. \square

We remark that statements similar to Proposition 1.1 and Corollary 1.2 hold (almost trivially) in the case of a bounded domain Ω with smooth boundary such that $a(x_o) > 0$ for some $x_o \in \Omega$.

We conclude this section by showing an application of the results obtained to the case of the Schrödinger operator $\Delta + \lambda a(x)$ on \mathbb{R}^m . We assume that the positive part $a_+(x)$ of $a(x)$ does not vanish identically, so that the results described above hold, and that it satisfies the estimate

$$a_+(x) \leq \frac{k}{|x|^2}$$

for some positive constant k . According to [BRS1] Lemma 2.3, if $A(t) \leq \frac{(m-2)^2}{4t^2}$, then the equation

$$\Delta \varphi + A(|x|) \varphi = 0$$

has a positive solution φ on \mathbb{R}^m . Thus, if $\lambda k \leq (m-2)^2/4$, then φ satisfies

$$\Delta \varphi + \lambda a(x) \varphi \leq 0,$$

which according to the above mentioned result of Fisher-Colbrie and Schoen, [FCS], gives $\lambda_1^{\Delta + \lambda a(x)}(M) \geq 0$. We conclude that λ_* is strictly positive, and, in fact,

$$(1.8) \quad \lambda_* \geq \frac{(m-2)^2}{4k}.$$

We will come back to this in Section 3 below.

2. EXISTENCE AND UNIQUENESS RESULTS

The established relationship between λ_* and $\lambda_1^{L_\mu}(M)$ allows us to apply to the present situation many of the results obtained in [BRS2]. In particular, we quote the following theorem which states the existence of minimal positive solutions of equation (0.1).

Theorem 2.1. *Let $a(x), b(x) \in C_{loc}^{0,\alpha}(M)$ for some $0 < \alpha < 1$. Assume that $b(x) > 0$, and that, having set $L = \Delta + a(x)$, we have*

$$\lambda_1^L(M) < 0.$$

Then the equation

$$(2.1) \quad \Delta u + a(x)u - b(x)u^\sigma = 0, \quad \sigma > 1,$$

has a unique minimal C^2 positive solution.

As mentioned in the introduction, if we assume that the function $a(x)$ is positive somewhere on M then the condition $\lambda_1^L(M) < 0$ amounts to the fact $\mu = 1$ is larger than the principal eigenvalue λ_* of the problem $\Delta u + \lambda a(x)u = 0$ on M , and Theorem 2.1 compares with the existence results in [AB], Theorem 2.2 and [DM2], Theorem 1 (for the latter, see also the remark after Theorem 2.3).

The proof of Theorem 2.1 uses the method of super- and sub-solutions, and the main task is the construction of a sub-solution, which is where the assumption on the sign of $\lambda_1^L(M)$ plays a crucial role. In the main result of this Section, Theorem 2.10 below, we describe conditions not expressed in terms of the sign of $\lambda_1^L(M)$, for which one can guarantee existence of a solution.

Our first result, Theorem 2.4 below, states that, if one has a global sub-solution of (2.1) and the set where the non-negative coefficient b is suitably small, it is always possible to prove the existence of a maximal solution.

The idea of the proof consists in applying the method of sub- and super-solutions to a sequence of boundary value problems on domains which exhaust the manifold.

Since a global sub-solution of (2.1) is given, one first needs to find local super-solution. If $b(x)$ is strictly positive a sufficiently large constant will do. Even if this is not the case, a super-solution can be found provided the set where $b(x)$ vanishes is small (see Theorem 2.3 below).

To apply the approximation method it is also crucial that the approximating sequence is monotonic, and this follows from the next comparison result.

Proposition 2.2. *Let $D \subset M$ be an open set with smooth boundary ∂D , and assume that $a(x), b(x)$ are functions in $C(\overline{D}) \cap C_{loc}^{0,\alpha}(D)$, $0 < \alpha < 1$, and that $b(x)$ is non-negative and does not vanish identically on any connected component of D . Let $u, v \in C^0(\overline{D}) \cap C^2(D)$ be a positive solutions on D of*

$$(2.2) \quad \Delta u + a(x)u - b(x)u^\sigma = 0$$

and

$$(2.3) \quad \Delta v + a(x)v - b(x)v^\sigma \leq 0$$

respectively. If $u \leq v$ on ∂D then $u \leq v$ on D .

Proof. Since $a(x)$ has indefinite sign, the standard comparison principle does not apply. To circumvent this problem, set $v_- = \delta u$ for some $\delta \in (0, 1]$. Since $b(x) \geq 0$

and $1 - \sigma < 0$ we have

$$\Delta v_- = \delta \Delta u \geq -a(x)v_- + b(x)\delta^{1-\sigma}v_-^\sigma \geq -a(x)v_- + b(x)v_-^\sigma.$$

Next let $v_+ = v$, so that $v_- \leq u \leq v_+$ on ∂D . By the monotone iteration scheme there exists a C^2 solution w of (2.2) with $w = u$ on ∂D , and $v_- \leq w \leq v_+ = v$ on D . In order to conclude it is enough to show that $w = u$, and to this end we apply an argument used in the proof of Lemma 2.2 in [BRS2], which we reproduce here for the sake of completeness and the convenience of the reader. Let Z be the vector field defined on D by the formula

$$Z = (w^2 - u^2)\nabla \log \frac{w}{u}.$$

Since u and w are solutions of (2.2), a direct calculation yields

$$\operatorname{div} Z = b(x)(w^2 - u^2)(w^{\sigma-1} - u^{\sigma-1}) + \left| \frac{w}{u} \nabla u - \nabla w \right|^2 + \left| \frac{u}{w} \nabla w - \nabla u \right|^2.$$

Integrating over D , applying the divergence theorem, and using the fact that $u \equiv w$ on ∂D yield

$$(2.4) \quad \int_D \left| \frac{w}{u} \nabla u - \nabla w \right|^2 + \left| \frac{u}{w} \nabla w - \nabla u \right|^2 = - \int_D b(x)(w^2 - u^2)(w^{\sigma-1} - u^{\sigma-1}) \leq 0.$$

It follows that $\nabla w - \frac{w}{u} \nabla u = 0$ so that $u = Bw$ on any connected component D_1 of D , for some constant $B > 0$. Inserting this into the inequality

$$- \int_{D_1} b(x)(w^2 - u^2)(w^{\sigma-1} - u^{\sigma-1}) \leq 0$$

yields

$$(1 - B^2)(1 - B^{\sigma-1}) \int_{D_1} b(x)w^{\sigma+1} \leq 0.$$

Since $w > 0$ and $b \geq 0$, $b \not\equiv 0$ on D_1 this forces $B = 1$. Thus $u = w$ on D_1 , and therefore $u = w$ on D , as required. \square

We remark that Proposition 2.2 holds if the coefficients $a(x)$ and $b(x)$ are only assumed to be continuous, and if the functions u and v are in $C^1(D) \cap C^0(\overline{D})$, provided we interpret (2.2) and (2.3) in weak sense. Under these weaker assumptions, the vector field Z will be only continuous, in general, but the proof may be carried out using a suitable version of the divergence theorem (see, e.g., [RS], pp. 477–478).

Before stating Theorem 2.4, we also need to make precise the sense in which the set where $b(x)$ vanishes is small.

Let $a(x) \in C^0(M)$ and let $L = \Delta + a(x)$. If Ω is a non-empty open set, the first Dirichlet eigenvalue $\lambda_1^L(\Omega)$ is variationally defined as in Section 1 by means of the formula

$$\lambda_1^L(\Omega) = \inf \left\{ \int_\Omega |\nabla \phi|^2 - a(x)\phi^2 : \phi \in H_o^1(\Omega), \int_\Omega \phi^2 = 1 \right\},$$

and, if Ω is bounded and both Ω and a are sufficiently regular, the infimum is attained and there exists a unique normalized eigenfunction v defined on Ω satisfying

$$\begin{cases} \Delta v + a(x)v + \lambda_1^L(\Omega)v = 0 & \text{on } \Omega \\ v > 0 & \text{on } \Omega, \quad v \equiv 0 \text{ on } \partial\Omega. \end{cases}$$

We extend the definition to an arbitrary bounded subset S of M , by setting

$$\lambda_1^L(S) = \sup \lambda_1^L(\Omega),$$

where the supremum is taken over all open bounded sets with smooth boundary Ω such that $S \subset \Omega$. Note that, by definition, if $S = \emptyset$ then $\lambda_1^L(S) = +\infty$. Finally, if S is an unbounded subset of M , we define

$$\lambda_1^L(S) = \inf \lambda_1(D \cap S),$$

where the infimum is taken over all bounded open sets with smooth boundary. Note that if $\{D_n\}$ is a increasing sequence of open sets with smooth boundary which exhausts M , then, by domain monotonicity, $\lambda_1^L(S) = \lim_n \lambda_1(D_n \cap S)$.

Since the first Dirichlet eigenvalue of the Laplacian of a ball B_r grows like r^{-2} as $r \rightarrow 0$, $\lambda_1^L(B_r) > 0$ provided r is sufficiently small, and one may think that the condition $\lambda_1^L(S) > 0$ expresses the fact that S is small in a spectral sense.

This notion of smallness is appropriate for our purposes. Indeed, P. Li, L.-F. Tam and D. Yang, [LYT], have established the following relationship between the first eigenvalue of the set where $b(x)$ vanishes, and the existence of a non trivial super-solution of equation (2.1).

Theorem 2.3. *Let $a(x)$ and $b(x)$ be Hölder continuous functions on M , with $b(x) \geq 0$ on M , and let $S_o = \{x \in M : b(x) = 0\}$. Let Ω be a bounded open domain in M , and let $L = \Delta + a(x)$. If equation (2.1) has a positive super-solution on Ω , then $\lambda_1^L(\Omega \cap S_o) \geq 0$.*

Conversely, if $\lambda_1^L(\Omega \cap S_o) > 0$ then (2.1) has a positive super-solution on Ω .

We remark that using Theorem 2.3 we may improve Theorem 2.1 above replacing the assumption that $b(x) > 0$ on M with the assumption that b is non-negative, and its zero set S_o is such that $\lambda_1^L(S_o) > 0$. Indeed, the strict positivity of b is only used to guarantee that on every bounded domain a suitably large constant is a super-solution of equation (2.1) (see [BRS2], p.184).

Note that, if we assume that S_o is a bounded domain with smooth boundary such that $a(x)$ is positive somewhere in S_o , then the condition $\lambda_1^L(S_o) > 0$ amounts to the fact that 1 is strictly smaller than the principal eigenvalue of the problem $\Delta u + \lambda a(x)u = 0$ with Dirichlet boundary conditions. Recalling the remark after the statement of Theorem 2.1 we conclude that if $\lambda_* < 1 < \lambda_1(L_\mu, S_o)$ then equation (2.1) has a unique positive minimal solution on M . This again compares with Theorem 1 in [DM2].

We are now ready to state

Theorem 2.4. *Let $a(x), b(x) \in C_{loc}^{0,\alpha}(M)$ for some $0 < \alpha \leq 1$. Assume that $b(x)$ is non-negative, and strictly positive off a compact set, and that, denoting with S_o the set where $b(x)$ vanishes, we have $\lambda_1^L(S_o) > 0$. If $u_- \in C^0(M) \cap H_{loc}^1(M)$, $u_- \geq 0$, $u_- \not\equiv 0$, is a global sub-solution of equation (2.1), then (2.1) has a maximal positive C^2 solution.*

Proof. Let D_k be an increasing exhaustion of M by open domain with smooth boundary, such that $S_o \subset D_k \subset \bar{D}_k \subset D_{k+1}$ for every k . Fix k in \mathbb{N} . Since $\lambda_1^L(S_o) > 0$, by Theorem 2.3 there exists a C^2 positive function v satisfying

$$\Delta v + a(x)v - b(x)v^\sigma \leq 0 \text{ on } D_{k+1}.$$

Since \bar{D}_k is compactly contained in D_{k+1} , $\inf_{D_k} v > 0$ and u_- is bounded on \bar{D}_k . Thus, given $n \geq \max_{\bar{D}_k} u_-$, there exists $C > 0$ large enough that the function

$v_+ = Cv$ satisfies

$$\begin{cases} \Delta v_+ + a(x)v_+ - b(x)v_+^\sigma \leq 0 & \text{on } D_k \\ v_+ \geq n \geq \max_{\bar{D}_k} u_- & \text{on } \partial D_k \text{ and } v_+ \geq u_- \text{ on } \bar{D}_k. \end{cases}$$

The monotone iteration scheme yields a solution $u_{k,n}$ of the boundary value problem

$$(2.5) \quad \begin{cases} \Delta u + a(x)u - b(x)u^\sigma = 0 & \text{on } B_k \\ u = n & \text{on } \partial B_k. \end{cases}$$

We now show that the sequence $\{u_{k,n}\}$ is uniformly bounded with respect to $n \in \mathbb{N}$ on compact subsets of D_k .

Assume first that K is a compact subset of D_k which does not intersect S_0 . Then we may find a positive constant b_o and a finite number of disjoint open balls B_i which cover K such that $b(x) \geq b_o$ on each B_i . Applying Lemma 2.6 in [PRS2] we deduce that there exists a constant $C_1 = C_1(K) > 0$ such that

$$(2.6) \quad u_{k,n}(x) \leq C_1 \quad \forall x \in K, \forall n.$$

Next we show that $u_{k,n}$ is uniformly bounded in a neighborhood of S_0 . By definition there exist open sets with smooth boundary Ω and Ω' such that $S_0 \subset \Omega \subset \bar{\Omega} \subset \Omega' \subset \bar{\Omega}' \subset D_k$ and $\lambda_1^L(\Omega') > 0$.

Note that since $\partial\Omega$ is a compact subset of D_k which does not intersect S_0 , there exists a constant C_2 such that $u_{k,n} \leq C_2$ on $\partial\Omega$. Next, let ϕ be a positive eigenfunction for L belonging to $\lambda_1^L(\Omega') > 0$. Since ϕ is positive on Ω' , it is bounded away from zero on $\bar{\Omega}$ and there exists a positive constant c such that $c\phi > C_2$ on $\bar{\Omega}$.

Note that

$$\Delta(c\phi) + a(x)(c\phi) = -\lambda_1^L(\Omega'')(c\phi) < 0,$$

while

$$\Delta u_{k,n} + a(x)u_{k,n} = b(x)u_{k,n}^\sigma \geq 0$$

on Ω , and $u_{k,n} \leq C_2 < c\phi$ on $\partial\Omega$,

We claim that $u_{k,n} \leq c\phi$ on Ω . Indeed, assume that this is not so, and let $A = \{x \in \Omega' : u_{k,n} - c\phi > 0\}$. Then A is non-empty and $\bar{A} \subset \Omega$, and we deduce that $w = u_{k,n} - c\phi$ attains a positive maximum in A . On the other hand w satisfies

$$\begin{cases} \Delta w + a(x)w \geq 0 & \text{in } A \\ w = 0 & \text{on } \partial A, \end{cases}$$

and therefore, by the generalized maximum principle, w/ϕ is constant on A , and since it vanishes on ∂A we conclude that $w/\phi = 0$ on A , that is, $w = 0$ on A , contradiction.

Thus $u_{k,n} \leq c\phi \leq C_2$ on $\bar{\Omega}$ and it follows easily that $u_{k,n}$ is uniformly bounded on compact subset of D_k .

By interior elliptic estimates, a subsequence of $u_{k,n}$ converges in C_{loc}^2 to a solution u_k^∞ of

$$(2.7) \quad \begin{cases} \Delta u + a(x)u - b(x)u^\sigma = 0 & \text{on } D_k \\ u = +\infty & \text{on } \partial D_k. \end{cases}$$

We consider the sequence $\{u_k^\infty\}$. Clearly, $u_k^\infty \geq u_- > 0$, and an exhaustion argument and Proposition 2.2 show that

$$(2.8) \quad u_{k+1}^\infty \leq u_k^\infty \quad \text{on } \overline{D}_k.$$

Since $\{u_k^\infty\}$ is monotone non-increasing, it converges to a function u which solves (2.1) and satisfies $u \geq u_- \geq 0$, $u_- \not\equiv 0$ on M . If u_1 is another positive solution of (2.1) on M , then $u_1 \leq u_k^\infty$ by Proposition 2.2, and therefore $u_1 \leq u$, thus proving the maximality of u . Finally, u is strictly positive for otherwise the non-negative function u_- would attain a zero minimum, thus violating the minimum principle ([GT], p. 35). \square

It is worth pointing out the following consequence of Proposition 2.2.

Proposition 2.5. *Let $a(x), b(x) \in C^0(M)$, $0 < \alpha < 1$, and $b(x) \geq 0$, $b(x) \not\equiv 0$. Then the problem*

$$(2.9) \quad \begin{cases} \text{(i)} & \Delta u + a(x)u - b(x)u^\sigma = 0, \quad \sigma > 1 \\ \text{(ii)} & \lim_{r(x) \rightarrow +\infty} u(x) = L > 0 \end{cases}$$

has at most one positive C^2 solution.

Proof. Let u, v be positive C^2 solutions of (2.9). Choose any $\epsilon > 0$ and observe that, since b is non-negative, the function $w_\epsilon = (1 + \epsilon)v$ is a super-solution of (2.9) (i) satisfying

$$(2.10) \quad \lim_{r(x) \rightarrow +\infty} w_\epsilon(x) = (1 + \epsilon)L.$$

Fix $R_0 > 0$ sufficiently large, so that for every $R > R_0$ we have $b(x) \not\equiv 0$ on B_R and $w_\epsilon - u > 0$ on ∂B_R . The latter is possible because of the limit relations (2.9) (ii) and (2.10). It follows from Proposition 2.2 that $w_\epsilon \geq u$ on B_R for every $R \geq R_0$. Thus, $u(x) \leq (1 + \epsilon)v(x)$ on M , and since $\epsilon > 0$ was arbitrary, $u \leq v$ on M . Interchanging u and v yields the reverse inequality, and equality follows. \square

We remark that the assumption $L > 0$ in the statement of the proposition cannot be weakened to $L \geq 0$. Indeed, in [BRS2], pp. 214–215, it is shown that on m -dimensional hyperbolic space \mathbb{H}^m , equation (2.9) with $a(x) \equiv m(m-2)/4$, $b(x) \equiv 1$ and $\sigma = (m+2)/(m-2)$ has a family of positive distinct radial solutions which tend to zero at infinity at the same rate.

It may also be worth noting that the assumptions on u and v in the uniqueness result obtained above may be weakened provided some conditions on the coefficients a and b and on the volume growth of the manifold are imposed.

Theorem 2.6. *Let $a(x), b(x) \in C^0(M)$ and assume that, for some $C > 0$ and $0 \leq \mu < 2$,*

$$(2.11) \quad \text{(i)} \quad b(x) \geq C(1 + r(x))^{-\mu} \quad \text{and} \quad \text{(ii)} \quad \sup_M \frac{a_-}{b} < +\infty.$$

Assume that u and v are C^2 nonnegative solutions of

$$\Delta u + a(x)u - b(x)u^\sigma \geq 0 \geq \Delta v + a(x)v - b(x)v^\sigma \quad \sigma > 1,$$

on M , satisfying

$$\liminf_{r(x) \rightarrow \infty} v(x) > 0, \quad \limsup_{r(x) \rightarrow \infty} u(x) < +\infty.$$

If

$$(2.12) \quad \liminf_{r \rightarrow +\infty} \frac{\log \text{vol } B_r}{r^{2-\mu}} < +\infty,$$

then $u \leq v$.

Proof. Note first of all that, by the maximum principle (see [GT], p. 35), v is strictly positive, and therefore, by the liminf condition, it is bounded away from 0 on M . Also, u is bounded above on M . We may assume that u is not identically zero, for else there is nothing to prove. Thus, $\beta = \sup_M \frac{u}{v}$ is finite and strictly positive. The conclusion of the theorem amounts to saying that $\beta \leq 1$. Assume by contradiction that $\beta > 1$, and let $\phi = u - \beta v$.

Clearly $\phi \leq 0$. We claim that $\sup_M \phi = 0$. Indeed, there exists a sequence x_n such that $\frac{u(x_n)}{v(x_n)} \rightarrow \beta > 0$, and since $u(x_n)$ is bounded above, so must be $v(x_n)$ (for else $\beta = 0$), and then

$$\phi(x_n) = v(x_n) \left(\frac{u(x_n)}{v(x_n)} - \beta \right) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

as claimed.

We write

$$(2.13) \quad \begin{aligned} \Delta \phi &\geq \Delta(u - \beta v) = -a(x)[u - \beta v] + b(x)[u^\sigma - \beta v^\sigma] \\ &= -a(x)\phi + b(x)[u^\sigma - (\beta v)^\sigma] + b(x)v^\sigma[\beta^\sigma - \beta]. \end{aligned}$$

By the mean value theorem we have

$$[u^\sigma - (\beta v)^\sigma](x) = h(x)(u - \beta v) = h(x)\phi$$

where

$$h(x) = \frac{\sigma}{u(x) - \beta v(x)} \int_{\beta v(x)}^{u(x)} t^{\sigma-1} dt$$

is continuous, and nonnegative on M . Further, since u is bounded above, it follows h is bounded above by a constant H on the set $\{x : \phi(x) > -1\}$. Also, since $\beta > 1$, $\sigma > 1$ and v is bounded away from zero,

$$v^\sigma[\beta^\sigma - \beta] \geq 2c > 0,$$

for some positive constant c . Inserting the above expressions in (2.13), noting that, since ϕ is non-positive, $-a(x)\phi \geq a_-(x)\phi$, and dividing through by $b(x)$, we obtain

$$\frac{1}{b} \Delta \phi \geq \left(\frac{a_-}{b} + h(x) \right) \phi + 2c.$$

Now let $\epsilon \in (0, 1)$ be such that

$$\left(\sup \frac{a_-}{b} + H \right) \epsilon < c,$$

and let $\Omega_\epsilon = \{x : \phi(x) > -\epsilon\}$, which is not empty since $\sup \phi = 0$. Then

$$\frac{1}{b(x)} \Delta \phi \geq c > 0 \quad \text{on } \Omega_\epsilon.$$

On the other hand, since the volume growth condition (2.12) holds, and $b(x)$ satisfies the lower estimate (2.11) (i), Theorem A in [PRS1] applies, and the weak maximum principle holds, namely,

$$\inf_{\Omega_\epsilon} \frac{1}{b(x)} \Delta \phi \leq 0$$

thus yielding the required contradiction. \square

As an immediate corollary we have

Corollary 2.7. *Let a and b satisfy the conditions listed in Theorem 2.6, and let u and v be nonnegative solutions of*

$$\Delta u + a(x)u - b(x)u^\sigma = 0.$$

If both u and v satisfy the condition

$$(2.14) \quad 0 < \liminf_{r(x) \rightarrow \infty} u(x) \leq \limsup_{r(x) \rightarrow \infty} u(x) < +\infty,$$

and (2.12) holds, then $u = v$.

As observed above, condition (2.14) amounts to requiring that u and v are bounded and bounded away from zero on M . We also note that the family of functions mentioned in the remark that follows Proposition 2.5 also shows that uniqueness fails if we do not assume that the liminf of u and v are strictly positive.

It is worth mentioning the following geometric consequence.

Corollary 2.8. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold of dimension $m \geq 3$ and scalar curvature $s(x)$ satisfying*

$$s(x) \leq -C(1 + r(x))^{-\mu}$$

for some constants $C > 0$ and $0 \leq \mu < 2$. Assume that (2.12) holds. Then any conformal diffeomorphism of M into itself which preserves the scalar curvature and whose stretching factor u satisfies (2.14), is an isometry.

Proof. Let $\phi : M \rightarrow M$ be a conformal diffeomorphism. Then $\phi^* \langle \cdot, \cdot \rangle = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle$ where u is the stretching factor. Since ϕ preserves the scalar curvature, u is a solution of

$$c_m \Delta u - s(x)u + s(x)u^{\frac{m+2}{m-2}} = 0,$$

(see, e.g., [PRS1], p. 1319 ff), and the result follows at once from Corollary 2.7. \square

We remark that here we require that u is bounded above and away from zero, so that the conformal diffeomorphism ϕ is a quasi-isometry. By contrast, in [PRS1] Corollary 3.4, ϕ is not assumed to be a quasi-isometry, but the scalar curvature $s(x)$ is assumed to be bounded below.

We now proceed with the main result of this section, Theorem 2.10, where we show that, under suitable assumptions on the coefficients, equation (2.1) has a globally defined positive sub-solution, and therefore, a maximal positive solution. The proof is based on the method of super and sub-solutions. This is achieved by constructing a sub-solution inside and outside a suitable ball in such a way that they can be glued together to yield a global sub-solution.

We begin with the following lemma, which will be the key ingredient in the construction of a sub-solution in the complement of a ball.

Lemma 2.9. *Let $A(r), B(r) \in C^0([0, +\infty))$ with $A(r) \geq 0$ and $B(r) > 0$ on $[0, +\infty)$. Let g be a non-decreasing smooth function on $[R, +\infty)$, for some $R > 0$. Then, given $T > 0$ and $\sigma > 1$, the problem*

$$(2.15) \quad \begin{cases} \alpha'' + (m-1)\frac{g'}{g}\alpha' - A(r)\alpha - B(r)\alpha^\sigma = 0 & \text{on } (R, R+T) \\ \alpha(R) = \alpha_o, \alpha(R+T) = 0 \end{cases}$$

has a C^2 solution α on $[R, R+T]$. Furthermore $\alpha > 0$ and $\alpha' < 0$ in $[R, R+T]$ and for every $T_o \in (0, T]$ the following estimate holds

$$(2.16) \quad |\alpha'(R)| \leq \left\{ \frac{g(R+T_o)}{g(R)} \right\}^{m-1} \left\{ T_o \max_{[R, R+T_o]} [A(s) + B(s)\alpha_o^{\sigma-1}] + \frac{1}{T_o} \right\} \alpha_o.$$

Proof. We first show the existence of a solution α to (2.15). We extend g to a smooth, non-decreasing function on $[0, +\infty)$ satisfying $g'(0) = 1$ and $g^{(2k)}(0) = 0$ for every $k \in \mathbb{N}$, and consider the model manifold $M = \mathbb{R}^m$ with the metric given in polar coordinates by

$$\langle \cdot, \cdot \rangle = dr^2 + g(r)^2 d\theta^2.$$

Note that by the conditions imposed on the function g , the metric originally defined on $\mathbb{R}^m \setminus \{0\}$ extends to a smooth metric on the whole of \mathbb{R}^m .

Let R_1 and \bar{R} be such that $0 < R_1 < R < R+T < \bar{R}$, and let ψ be a smooth radial cut-off function such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ on B_{R_1} , and $\psi \equiv 0$ on $M \setminus \bar{B}_{R+T}$. Define

$$\bar{a}(x) = -\psi(x)A(r(x)) + N(1 - \psi(x))$$

where N is constant and let \bar{L} be the Schrödinger operator $L = \Delta + \bar{a}(x)$. We claim that if N is sufficiently large then

$$\lambda_1^{\bar{L}}(B_{\bar{R}}) < 0.$$

Indeed, let u be any smooth function satisfying $u > 0$ in $B_{\bar{R}}$ and $u = 0$ on $\partial B_{\bar{R}}$. Then

$$\int_{B_{\bar{R}}} |\nabla u|^2 - \bar{a}(x)u^2 = \int_{B_{\bar{R}}} [|\nabla u|^2 + \psi(x)A(r(x))u^2] - N \int_{B_{\bar{R}}} (1 - \psi)u^2,$$

and since $(1 - \psi)u^2 > 0$ in $B_{\bar{R}} \setminus B_{R+T}$, the right hand side may be made negative provided N is large enough. The claim now follows from the variational characterization of $\lambda_1^{\bar{L}}(B_{\bar{R}})$.

Let ϕ be the radial, normalized eigenfunction belonging to $\lambda_1^{\bar{L}}(B_{\bar{R}})$. By definition

$$\int_{B_{\bar{R}}} |\nabla \phi|^2 - \bar{a}(x)\phi^2 = \lambda_1^{\bar{L}}(B_{\bar{R}}) < 0,$$

so that, if $\gamma > 0$ is sufficiently small,

$$\int_{B_{\bar{R}}} |\nabla \phi|^2 - \bar{a}(x)\phi^2 + \gamma B(r(x))\phi^2 = \lambda_1^{\bar{L}}(B_{\bar{R}}) + \gamma \int_{B_{\bar{R}}} B(r(x))\phi^2 < 0.$$

Thus, if we denote by $\tilde{L} = L + \bar{a}(x) - \gamma B(r(x))$, then

$$\lambda_1^{\tilde{L}}(B_{\bar{R}}) < 0.$$

If ξ is a positive radial eigenfunction belonging to $\lambda_1^{\tilde{L}}(B_{\bar{R}})$, so that

$$\Delta \xi + \bar{a}(x)\xi = \gamma B(r(x))\xi - \lambda_1^{\tilde{L}}(B_{\bar{R}})\xi \geq \gamma B(r(x))\xi \quad \text{on } B_{\bar{R}},$$

then the function $v_- = \mu\xi$ satisfies

$$\begin{cases} \Delta v_- + \bar{a}(x)v_- - B(r(x))v_-^\sigma \geq B(r(x))\mu\xi[\gamma - (\mu\xi)^{\sigma-1}] \geq 0 & \text{in } B_{\bar{R}} \\ v_- > 0 & \text{on } B_{\bar{R}} \quad v_- \equiv 0 & \text{on } \partial B_{\bar{R}} \end{cases}$$

provided $0 < \mu < \gamma^{1/(\sigma-1)}(\sup_{B_{\bar{R}}} \xi)^{-1}$.

On the other hand, since $B(r) > 0$, a sufficiently large positive constant v_+ is a super solution of the above problem, that is

$$\begin{cases} \Delta v_+ \bar{a}(x) v_+ - B(r(x)) v_+^\sigma \leq 0 & \text{in } B_{\bar{R}} \\ v_+ > 0 & \text{on } \partial B_{\bar{R}}, \end{cases}$$

and by the monotone iteration scheme there exists a solution w to the problem

$$\begin{cases} \Delta w + \bar{a}(x) w - B(r(x)) w^\sigma = 0, & w > 0 & \text{in } B_{\bar{R}} \\ w \equiv 0 & \text{on } \partial B_{\bar{R}}. \end{cases}$$

Note that w is radial, since the monotone iteration scheme produces radial solutions in radial setting. Moreover, since $\bar{a}(x) > -A(r(x))$ and $\inf_{\bar{B}_{R+T}} w > 0$, if c is sufficiently large then the function $w_+ = cw$ satisfies

$$\begin{cases} \Delta w_+ - A(r(x)) w_+ - B(r(x)) w_+^\sigma \leq 0 & \text{in } B_{\bar{R}} \\ w_+ \geq 0 & \text{on } \partial B_{R+T} \cup \partial B_R, \end{cases}$$

and since $w_- \equiv 0$ is a sub-solution of the problem, applying once again the monotone iteration scheme produces a radial non-negative C^2 solution u of

$$\begin{cases} \Delta u + -A(r(x)) u - B(r(x)) u^\sigma = 0 & \text{in } B_{R+T} \setminus \bar{B}_R \\ u \geq 0, u = \alpha_o & \text{on } \partial B_R, u = 0 & \text{on } \partial B_{R+T}. \end{cases}$$

Since u satisfies

$$\Delta u - (A(r(x)) + B(r(x)) u^{\sigma-1}) u = 0,$$

by the strong maximum principle (see [GT], p. 35) we deduce that $u > 0$ in $B_{R+T} \setminus \bar{B}_R$. Since u is radial, we deduce that $u(x) = \alpha(r(x))$ with α satisfying (2.15).

We now show that $\alpha' < 0$ in $[R, R+T)$. Indeed, assume this is not so. Since $\alpha(r) > 0 = \alpha(R+T)$, then there exists $r_o \in [R, R+T)$ such that $\alpha'(r_o) = 0$. Integrating (2.15) between r_o and r gives

$$g(r)^{m-1} \alpha'(r) = \int_{r_o}^r g(s)^{m-1} \alpha(s) [A(s) + B(s) \alpha(s)^{\sigma-1}] ds$$

and the integral on the right hand side is strictly positive for every $r > r_o$. Thus $\alpha'(r) > 0$ in $[r_o, R+T]$, contradicting $\alpha(r_o) > 0$, $\alpha(R+T) = 0$.

It remains to show that estimate (2.16) holds. Integrating (2.15) between $R+t_1$ and $R+t_2$ with $0 \leq t_1 < t_2 \leq T$, yields

$$\begin{aligned} (2.17) \quad & g(R+t_2)^{m-1} \alpha'(R+t_2) - g(R+t_1)^{m-1} \alpha'(R+t_1) \\ &= \int_{R+t_1}^{R+t_2} g(s)^{m-1} \alpha(s) [A(s) + B(s) \alpha(s)^{\sigma-1}] ds, \end{aligned}$$

and since the integrand is positive, we deduce that

$$g(R+t_1)^{m-1} \alpha'(R+t_1) < g(R+t_2)^{m-1} \alpha'(R+t_2),$$

whence, recalling that $\alpha' < 0$ and g is non-decreasing,

$$\frac{\alpha'(R+t_1)}{\alpha'(R+t_2)} > \left(\frac{g(R+t_2)}{g(R+t_1)} \right)^{m-1} \geq 1,$$

and therefore

$$\alpha'(R+t_1) < \alpha'(R+t_2) \leq 0 \quad \forall 0 \leq t_1 < t_2 \leq T.$$

Moreover, for every $T_o \in (0, T]$ there exists $t \in (0, T_o)$ such that

$$\frac{\alpha_o}{T_o} = \frac{\alpha(R) - \alpha(R+T)}{T_o} = -\alpha'(R+t) > -\alpha(R+T_o).$$

Writing (2.17) with $t_1 = 0$ $t_2 = T_o$, and using the above inequality we obtain

$$-g(R)^{m-1}\alpha'(R) \leq g(R+T_o)^{m-1}\frac{\alpha_o}{T_o} + T_o g(R+T_o)^{m-1}\alpha_o \max_{[R, R+T_o]} [A(s) + B(s)\alpha_o^{\sigma-1}],$$

whence, for every $t \in [R, R+T]$,

$$|\alpha'(t)| \leq |\alpha'(R)| \leq \left(\frac{g(R+T_o)}{g(R)} \right)^{m-1} \left\{ \frac{1}{T_o} + T_o \max_{[R, R+T_o]} [A(s) + B(s)\alpha_o^{\sigma-1}] \right\} \alpha_o,$$

as required. \square

We are now ready to prove the main result of this section.

Theorem 2.10. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete manifold, and assume that the differential inequality*

$$(2.18) \quad \Delta r(x) \leq (m-1) \frac{g'(r(x))}{g(r(x))}$$

holds pointwise in the complement of the cut locus Cut_o of o , for some function $g \in C^\infty([0, +\infty))$, with $g^{(2k)}(0) = 0$ if $k = 0, 1, 2, \dots$, $g'(0) = 1$, $g(t) > 0$ and $g'(t) \geq 0$ if $t > 0$. Let $a(x), b(x) \in C_{loc}^{0,\alpha}(M)$ ($0 < \alpha \leq 1$), and assume that $b(x)$ satisfies the conditions in Theorem 2.4. Suppose that we can choose $R > 0$ and $T_o > 0$ in such a way that

$$(2.19) \quad R \inf_{B_R} a > (1 + \sup_{B_R} r \Delta r) \left(\frac{1}{T_o} + T_o \max_{\overline{B_{R+T_o}} \setminus B_R} a_- \right) \left(\frac{g(R+T_o)}{g(R)} \right)^{m-1}.$$

Then the equation

$$(2.20) \quad \Delta u + a(x)u - b(x)u^\sigma = 0, \quad \sigma > 1,$$

has a maximal positive solution on M .

Remarks Observe that, by the Laplacian Comparison Theorem, see, e.g. [GW], or [BRS2], Appendix, the validity of an inequality of the form (2.18) can be deduced from appropriate lower bounds for the radial Ricci curvature. To compare with the existence theorems in Euclidean setting that can be found in the literature, we note that, on \mathbb{R}^m , (2.18) holds with $g(r) = r$, while if we assume an "almost Euclidean behavior", namely, that the radial Ricci curvature satisfies an estimate

$$Ricc \geq -(m-1)B(1+r^2)^{-1},$$

then (2.18) holds with $g(r) = r^{B'}$ where $B' = [1 + \sqrt{1 + 4B^2}]/2$. We note in passing that the assumption on the Ricci curvature does not imply that the manifold is quasi-isometric to Euclidean space. If we assume instead that

$$Ricc \geq -(m-1)B$$

so that, loosely speaking, the reference model is hyperbolic space of constant negative sectional curvature, then (2.18) holds with the choice $g(r) = \frac{1}{\sqrt{B}} \sinh(\sqrt{B}r)$.

We also note that condition (2.19) is satisfied if $a(x)$ is sufficiently large near the origin o and non-negative in a suitably large ball. Further, up to choosing a different reference point, it suffices to assume that $a(x)$ has a positive spike somewhere. Of

course (2.18) should then be written with respect to the new origin centered at the spike. Observe however that if we assume, e.g., that the Ricci curvature is bounded from below by a negative constant, then the validity of (2.18) is independent of the chosen origin.

Proof. According to Theorem 2.4 it suffices to show that (2.20) has a non-negative, non-identically zero global sub-solution. This will be obtained by joining suitable radial local sub-solutions. We set

$$\min_{\partial B_r} a(x) = A(r) = A_+(r) - A_-(r), \quad B(r) = \max_{\partial B_r} b(x), \quad \text{and} \quad \tau = \max_{B_r} r \Delta r.$$

Applying the previous lemma with $A_-(r)$ and $B(r) + \epsilon$ instead of $A(r)$ and $B(r)$, respectively, we deduce that for every α_o and $\epsilon > 0$ there exists a solution $\alpha \in C^2([R, R + T_o])$ of differential inequality

$$(2.21) \quad \alpha'' + (m-1) \frac{g'}{g} \alpha' - A_-(r) \alpha - B(r) \alpha^\sigma \geq 0,$$

satisfying $\alpha(R) = \alpha_o$, $\alpha(R + T_o) = 0$, $\alpha'(r) < 0$ on $[R, R + T_o]$ and

$$(2.22) \quad 0 > \alpha'(R) > - \left(\frac{g(R + T_o)}{g(R)} \right)^{m-1} \left\{ \frac{1}{T_o} + T_o \max_{[R, R+T_o]} [A(s) + (B(s) + \epsilon) \alpha_o^{\sigma-1}] \right\} \alpha_o.$$

Using the expression of the Laplacian of a radial function, and the inequalities (2.18), $a(x) \geq -A_-(r(x))$ and $b(x) \leq B(r(x))$ and $\alpha' \leq 0$, it follows that the function $v(x) = \alpha(r(x))$ is Lipschitz in $B_{R+T_o} \setminus B_R$ and C^2 in the complement of the cut locus of o where it satisfies the pointwise inequality

$$(2.23) \quad \Delta v + a(x)v - b(x)v^\sigma = \alpha'' + \alpha' \Delta r + a(x)\alpha - b(x)\alpha^\sigma \geq 0.$$

To construct a sub-solution in the ball B_R we consider the problem

$$(2.24) \quad \begin{cases} \beta'' + \frac{\tau}{r} \beta' + A(r)\beta - B(r)\beta^\sigma \geq 0 & \text{on } [0, R] \\ \beta > 0, \beta' \leq 0, \beta(R) = \alpha_o, \beta'(R) \leq \alpha'(R), \end{cases}$$

and we look for a solution of the form

$$(2.25) \quad \beta(r) = \alpha_o(1 + (R^2 - r^2)\eta) \geq \alpha_o.$$

Note that $\beta'(r) = -2\eta R \alpha_o < 0$, so, using (2.22), the condition $\beta'(R) \leq \alpha'(R)$ follows from

$$(2.26) \quad 2\eta \geq \frac{1}{R} \left(\frac{g(R + T_o)}{g(R)} \right)^{m-1} \left\{ \frac{1}{T_o} + T_o \max_{[R, R+T_o]} [A(s) + (B(s) + \epsilon) \alpha_o^{\sigma-1}] \right\}.$$

On the other hand, a direct computation that uses $A(r) \geq \min_{B_R} a(x)$ and $B(r) \leq \max_{B_R} b(x)$ shows that

$$\beta'' + \frac{\tau}{r} \beta' + A(r)\beta - B(r)\beta^\sigma \geq \alpha_o \left\{ \min_{B_R} a(x) - 2\eta(1 + \tau) - \alpha_o^{\sigma-1} \eta^\sigma R^\sigma \max_{B_r} b(x) \right\}$$

and the right hand side is non-negative provided

$$(2.27) \quad 2\eta \leq \frac{1}{1 + \tau} \left\{ \min_{B_R} a - \alpha_o^{\sigma-1} \eta^\sigma R^\sigma \max_{B_r} b(x) \right\}.$$

Using (2.19) we may choose α_o small enough that, for every $\eta \leq (\min_{B_R} a)/[2(1 + \tau)]$ (which is the largest possible value of η allowed by (2.27)) the right hand side of (2.27) is greater than or equal to the right hand side of (2.26), and therefore choose

η in such a way that both (2.26) and (2.27) are satisfied. For such values of α_o and η , the function β satisfies all the requirements. Proceeding as above one verifies that the function $w(x) = \beta(\alpha(r(x)))$ is Lipschitz in B_R and C^2 in the complement of the cut locus where it satisfies the pointwise inequality

$$(2.28) \quad \Delta w + a(x)w - b(x)w^\sigma \geq 0.$$

Now we define

$$u_-(x) = \begin{cases} w & \text{on } B_R \\ v & \text{on } B_{R+T_o} \setminus B_R \\ 0 & \text{on } M \setminus B_{R+T_o}, \end{cases}$$

and claim that $u_- \in C^0(M) \cap H_{loc}^1(M)$ is a weak global sub-solution of (2.20).

This is easily seen if we assume that o is a pole of M , for then, given a positive test function $\varphi \in C_c^\infty(M)$, applying Green's second identity, and using the fact that w and v are pointwise sub-solutions of (2.20) in B_R and $B_{R+T_o} \setminus \overline{B_R}$ respectively, we obtain

$$\begin{aligned} \int_M u_- \Delta \varphi &= \int_M \langle \nabla u_-, \nabla \varphi \rangle = \int_{B_R} \varphi \Delta w + \int_{B_{R+T_o} \setminus B_R} \varphi \Delta v \\ &+ \int_{\partial B_R} (w - v) \langle \nabla \varphi, \nabla r \rangle - \varphi \langle \nabla(w - v), \nabla r \rangle + \int_{\partial B_{R+T_o}} v \langle \nabla \varphi, \nabla r \rangle - \varphi \langle \nabla v, \nabla r \rangle \\ &\geq \int_M (-a(x)u_- + b(x)u_-^\sigma) \varphi - (\beta'(R) - \alpha'(R)) \int_{\partial B_R} \varphi - \alpha'(R + T_o) \int_{\partial B_{R+T_o}} \varphi, \end{aligned}$$

and the claim follows from the inequalities $\beta'(R) \leq \alpha'(R)$, $\alpha'(R + T_o) \leq 0$.

In the case where the cut locus of o is not empty, one can adapt an argument in [PRS2], Lemma 2.2, as follows: we consider an exhaustion Ω_n of $M \setminus \text{Cut}_o$ by domains with smooth boundary, which are star shaped with respect to o , so that, denoting by ν the outward unit normal, we have $\langle \nabla r, \nu \rangle > 0$ on $\partial \Omega_n$. Since the part of ∂B_R contained in Ω_n is smooth, we may also deform Ω_n , if necessary, in such a way that, for every n , $\partial \Omega_n$ is transversal to ∂B_R and ∂B_{R+T_o} .

Since ∇u_- is locally bounded we have

$$\int_M u_- \Delta \varphi = - \int_M \langle \nabla u_-, \nabla \varphi \rangle = - \lim_{n \rightarrow \infty} \int_{\Omega_n} \langle \nabla u_-, \nabla \varphi \rangle.$$

We write $\Omega_n = (\Omega_n \cap B_R) \cup (\Omega_n \cap (B_{R+T_o} \setminus B_R)) \cup (\Omega_n \cap B_{R+T_o}^c)$, apply the divergence theorem, use the fact that v and w are pointwise sub-solutions of (2.20) in the complement of the cut locus, to obtain

$$\begin{aligned} - \int_{\Omega_n} \langle \nabla u_-, \nabla \varphi \rangle &\geq \int_{\Omega_n} \varphi [-a(x)u_- + b(x)u_-^\sigma] \\ &- \left(\int_{\partial(\Omega_n \cap B_R)} \varphi \langle \nabla w, \nu \rangle + \int_{\partial(\Omega_n \cap (B_{R+T_o} \setminus B_R))} \varphi \langle \nabla v, \nu \rangle \right). \end{aligned}$$

To conclude, note that, by the transversality assumption, up to sets of lower dimension, we have $\partial(\Omega_n \cap B_R) = (\Omega_n \cap \partial B_R) \cup (\partial \Omega_n \cap B_R)$ and similarly when B_R

is replaced by $B_{R+T_o} \setminus B_R$, so that the boundary integrals become

$$\begin{aligned} (\beta'(R) - \alpha'(R)) \int_{\Omega_n \cap \partial B_R} \varphi + \alpha'(R) \int_{\Omega_n \cap \partial B_{R+T_o}} \varphi \\ + \int_{\partial \Omega_n \cap B_R} \varphi \beta' \langle \nabla r, \nu \rangle + \int_{\partial \Omega_n \cap (B_{R+T_o} \setminus B_R)} \varphi \alpha' \langle \nabla r, \nu \rangle \end{aligned}$$

and the first integral is non-positive because $\beta'(R) \leq \alpha'(R)$, while the last two are non-positive because $\alpha', \beta' \leq 0$ and $\langle \nabla r, \nu \rangle \geq 0$ on $\partial \Omega_n$. \square

Remark 2.11. The above existence result is also relevant to the Yamabe problem, that is, the possibility of conformally deforming the assigned metric, with scalar curvature $s(x)$, to a new one with prescribed scalar curvature $K(x)$. Indeed assume that $m \geq 3$, and denote by $u^{2/(m-2)}$ the conformal factor, so that the deformed metric is given by $\langle \cdot, \cdot \rangle = u^{\frac{2}{m-2}} \langle \cdot, \cdot \rangle$. Then the scalar curvature of $\langle \cdot, \cdot \rangle$ is $K(x)$ provided u is a, necessarily positive, solution of the Yamabe equation

$$c_m \Delta u - s(x)u + K(x)u^{\frac{m+2}{m-2}} = 0$$

where $c_m = \frac{4(m-1)}{m-2}$, which is (2.20) with $a(x) = -c_m^{-1}s(x)$ and $b(x) = -c_m^{-1}K(x)$.

By way of example, assume that $\text{Ric} \geq -(m-1)B$, so that $s(x) \geq -m(m-1)B$, and, as noted above, (2.18) holds with $g(r) = (\sqrt{B})^{-1} \sinh(\sqrt{B}r)$. If we suppose that $s(x) \leq 0$ in a ball of radius $R+1$, then (2.19) holds, with $T_o = 1$, provided

$$R \sup_{B_R} s(x) \leq -c_m \left(1 + (m-1)R\sqrt{B} \coth(\sqrt{B}R) \right) \left(\frac{\sinh(\sqrt{B}(R+1))}{\sinh(\sqrt{B}R)} \right)^{m-1}.$$

It is easy to see that if $\sqrt{B} > 4(m-1)/[m(m-2)]$ then the above condition is verified provided $s(x)$ is sufficiently near $-m(m-1)B$ in the ball B_R and R is large enough. We note that there are situations (for instance when the sectional curvature is suitably pinched) where Theorem 2.10 is applicable, while Theorem 2.1 is not.

3. NON EXISTENCE RESULTS

The purpose of this section is to prove a non-existence result for non-negative C^2 solutions of the differential inequality

$$(3.1) \quad \Delta u + a(x)u - b(x)u^\sigma \geq 0 \quad \text{on } M.$$

We begin with the following general

Theorem 3.1. *Let $a(x), b(x) \in C^0(M)$ and assume that $b(x) \geq 0$. Let $H > 0$, $K > -1$ and $A \in \mathbb{R}$ be constants satisfying*

$$(3.2) \quad \max\{0, A\} \leq H(K+1) - 1,$$

and suppose that there exists a positive C^2 solution of the differential inequality

$$(3.3) \quad \Delta \varphi + Ha(x)\varphi \leq -K \frac{|\nabla \varphi|^2}{\varphi} \quad \text{on } M.$$

Then the differential inequality

$$(3.4) \quad u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2, \quad \sigma \geq 1,$$

has no non-negative C^2 solutions on M satisfying

$$(3.5) \quad \text{supp } u \cap \{x \in M : b(x) > 0\} \neq \emptyset$$

and

$$(3.6) \quad \left(\int_{\partial B_r} \varphi^{\frac{\beta+1}{H}(2-p)} u^{2(\beta+1)} \right)^{-1} \notin L^1(+\infty),$$

for some $p > 1$ and β satisfying $\max\{0, A\} \leq \beta \leq H(K+1) - 1$.

Remarks. We note that if $p = 2$, then the non-integrability assumption (3.6) involves u alone and reduces to

$$(3.7) \quad \left(\int_{\partial B_r} u^{2(\beta+1)} \right)^{-1} \notin L^1(+\infty).$$

If $p \neq 2$, assumption (3.7) implies (3.6) if a suitable bound on φ is available, e.g., if $p < 2$ and φ is bounded from above. A similar simplification occurs if φ is bounded above, respectively below, by a radial function, see, e.g., Theorem 3.10 below.

We also remark that the inequality

$$r - R = \int_R^r f^{1/2} f^{-1/2} \leq \left(\int_R^r f \right)^{1/2} \left(\int_R^r f^{-1} \right)^{1/2}$$

valid for $f > 0$, together with integration in polar coordinates, shows that condition (3.6) is implied by $\varphi^{\frac{2-p}{H}} u \in L^{2(\beta+1)}(M)$.

Finally, we note that the proof of the theorem could be achieved by adapting the argument in the proof of Theorem 1.3 in [PRS3]. However, the present assumptions allow us to give an alternative argument that we describe here for the sake of completeness.

Proof. Let $u \geq 0$ be a solution of (3.4) on M satisfying (3.5) and (3.6). Fix $\epsilon > 0$, $\alpha \in \mathbb{R}$ and set

$$v = \varphi^{-\alpha} (u^2 + \epsilon)^{(\beta+1)/2}.$$

A straightforward computation that uses (3.3) and (3.4) yields

$$\begin{aligned} \text{vdiv}(\varphi^{2\alpha} \nabla v) &\geq \alpha(K - \alpha + 1)(u^2 + \epsilon)^{\beta+1} \frac{|\nabla \varphi|^2}{\varphi^2} + (\beta + 1)(u^2 + \epsilon)^\beta b(x) u^{\sigma+1} \\ &\quad + a(x)(u^2 + \epsilon)^{\beta+1} \left[\alpha H - (\beta + 1) \frac{u^2}{u^2 + \epsilon} \right] \\ &\quad + (\beta + 1)(u^2 + \epsilon)^\beta \left[1 - A + (\beta - 1) \frac{u^2}{u^2 + \epsilon} \right] |\nabla u|^2. \end{aligned}$$

We choose $\alpha = H^{-1}(\beta + 1)$, so that our assumptions on β , H and K yield $0 < \alpha \leq K + 1$. Therefore $\alpha(K - \alpha + 1) \geq 0$, and using the assumptions $b(x) \geq 0$ and $\beta + 1 \geq 0$, we deduce that

$$(3.8) \quad \begin{aligned} \text{vdiv}(\varphi^{2\alpha} \nabla v) &\geq \epsilon(\beta + 1)a(x)(u^2 + \epsilon)^\beta \\ &\quad + (\beta + 1)(u^2 + \epsilon)^\beta \left[1 - A + (\beta - 1) \frac{u^2}{u^2 + \epsilon} \right] |\nabla u|^2. \end{aligned}$$

Let $r(t) \in C^1(\mathbb{R})$ and $s(t) \in C^0(\mathbb{R})$ satisfy the conditions

$$(3.9) \quad (i) \quad r(v) \geq 0, \quad r(v) + vr'(v) \geq s(v) > 0, \quad \text{on } [0, +\infty),$$

and let Z be the vector field defined by $Z = vr(v)\varphi^{2\alpha}\nabla v$. For fixed t and $\delta > 0$ let also ψ_δ be the Lipschitz function defined by

$$\psi_\delta(x) = \begin{cases} 1 & \text{if } r(x) \leq t \\ \frac{t + \delta - r(x)}{\delta} & \text{if } t < r(x) < t + \delta \\ 0 & \text{if } r(x) \geq t + \delta. \end{cases}$$

Using (3.8) (3.9) and the definition of ψ_δ we compute

$$\begin{aligned} \operatorname{div}(\psi_\delta Z) &= \psi_\delta \operatorname{div} Z + \langle \nabla \psi_\delta, Z \rangle \\ &\geq (\beta + 1)r(v)(u^2 + \epsilon)^\beta (\epsilon a(x) + [1 - A + (\beta - 1)\frac{u^2}{u^2 + \epsilon}]) |\nabla u|^2 \chi_{B_t} \\ &\quad + s(v)\varphi^{2\alpha} |\nabla v|^2 \chi_{B_t} + \frac{1}{\delta} \langle \nabla r, Z \rangle \chi_{\overline{B}_{t+\delta} \setminus B(t)}, \end{aligned}$$

whence, integrating, and using the divergence theorem and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_{B_t} (\beta + 1)r(v)(u^2 + \epsilon)^\beta (\epsilon a(x) + [1 - A + (\beta - 1)\frac{u^2}{u^2 + \epsilon}]) |\nabla u|^2 \\ + \int_{B_t} s(v)\varphi^{2\alpha} |\nabla v|^2 \leq \frac{1}{\delta} \int_{\overline{B}_{t+\delta} \setminus B(t)} |Z|. \end{aligned}$$

By Hölder inequality the integral on the right hand side is bounded above by

$$\left(\frac{1}{\delta} \int_{\overline{B}_{t+\delta} \setminus B(t)} \varphi^{2\alpha} \frac{r(v)^2}{s(v)} v^2 \right)^{1/2} \left(\frac{1}{\delta} \int_{\overline{B}_{t+\delta} \setminus B(t)} \varphi^{2\alpha} s(v) |\nabla v|^2 \right)^{1/2}.$$

Inserting into the above inequality, letting $\delta \rightarrow 0+$ and using the co-area formula (see Theorem 3.2.12 in [F]) we deduce that

$$\begin{aligned} (3.10) \quad \int_{B_t} (\beta + 1)r(v)(u^2 + \epsilon)^\beta (\epsilon a(x) + [1 - A + (\beta - 1)\frac{u^2}{u^2 + \epsilon}]) |\nabla u|^2 + s(v)\varphi^{2\alpha} |\nabla v|^2 \\ \leq \left(\int_{\partial B_t} \varphi^{2\alpha} \frac{r(v)^2}{s(v)} v^2 \right)^{1/2} \left(\int_{\partial B_t} \varphi^{2\alpha} s(v) |\nabla v|^2 \right)^{1/2}. \end{aligned}$$

In the above formula, the surface integral is computed with respect to $(m-1)$ -dimensional Hausdorff measure on ∂B_t , which coincides with the Riemannian measure induced on the regular part of ∂B_t (the intersection of ∂B_t with the complement of the cut locus of o , see [F], 3.2.46, or [Ch], Proposition 3.4). As $\epsilon \rightarrow 0$, $v = v_\epsilon \rightarrow v_0 = \varphi^{-\alpha} u^{\beta+1}$, whence, using the dominated convergence theorem in (3.10), we get

$$\begin{aligned} (3.11) \quad \int_{B_t} s(v_0)\varphi^{2\alpha} |\nabla v_0|^2 + (\beta + 1)(\beta - A) \int_{B_t} r(v_0)u^{2\beta} |\nabla u|^2 \\ \leq \left(\int_{\partial B_t} \varphi^{2\alpha} \frac{r(v_0)^2}{s(v_0)} v_0^2 \right)^{1/2} \left(\int_{\partial B_t} \varphi^{2\alpha} s(v_0) |\nabla v_0|^2 \right)^{1/2}. \end{aligned}$$

Defining

$$h(t) = \int_{B_t} \varphi^{2\alpha} s(v_0) |\nabla v_0|^2,$$

so that by the co-area formula H is Lipschitz and

$$h'(t) = \int_{\partial B_t} \varphi^{2\alpha} s(v_0) |\nabla v_0|^2,$$

and noting that the coefficient of the second integral on the left hand side of (3.11) is non-negative by the conditions imposed on β , we obtain

$$(3.12) \quad h(t) \leq \left(\int_{\partial B_t} \varphi^{2\alpha} \frac{r(v_0)^2}{s(v_0)} v_0^2 \right)^{1/2} (h'(t))^{1/2}$$

Our aim is to show that under assumption (3.6) v_0 is constant. The proof follows the lines of that of Lemma 1.1 in [RS]. Assume by contradiction that v_0 is not constant. Then there exists R_o such that $h(t) > 0$ for every $t \geq R_o$, and therefore the right hand side of (3.12) is positive for $t \geq R_o$. Dividing through by $h(t)$, squaring and integrating the resulting differential inequality between R and r with $R_o \leq R < r$ yield

$$(3.13) \quad h(R)^{-1} \geq h(R)^{-1} - h(r)^{-1} \geq \int_R^r \left(\int_{\partial B_t} \varphi^{2\alpha} \frac{r(v_0)^2}{s(v_0)} v_0^2 \right)^{-1} dt.$$

We choose a sequence of functions

$$r_n(t) = (t^2 + \frac{1}{n})^{\frac{p-2}{2}}, \quad s_n(t) = \min\{p-1, 1\} r_n(t), \quad \forall n \in \mathbb{N}, \quad p > 1.$$

Since condition (3.9) holds for every n , so does (3.13), whence, letting $n \rightarrow +\infty$ and using the Lebesgue and monotone convergence theorems we deduce that there exists $C > 0$ which depends only on p such that

$$(3.14) \quad \left(\int_{B_R} v_0^{p-2} \varphi^{2\alpha} |\nabla v_0|^2 \right)^{-1} \geq C \int_R^r \left(\int_{\partial B_t} \varphi^{2\alpha} v_0^p \right)^{-1} dt.$$

Now, recalling that $\alpha = (\beta + 1)/H$ and the definition of v_0 , we have $\varphi^{2\alpha} v_0^p = \varphi^{(2-p)(\beta+1)/H} u^{2(\beta+1)}$ and the required contradiction is reached by letting $r \rightarrow +\infty$ and using assumption (3.6).

Thus v_0 is constant, and we deduce that there exists a constant $C \geq 0$ such that

$$u^H = C\varphi.$$

Since u is not identically zero by (3.5), $C > 0$ and u is strictly positive on M . We insert the expression of φ in terms of u in (3.3), divide by CHu^{H-2} and subtract the result from (3.4) to obtain

$$[A - H(K+1) + 1] |\nabla u|^2 \geq b(x) u^{\sigma+1}.$$

Since the coefficient of $|\nabla u|^2$ is non-positive, by (3.2), we conclude that

$$b(x) u^{\sigma+1} \leq 0,$$

which contradicts (3.5). □

Remark 3.2. Observe that the above proof actually shows that if $A < H(K+1) - 1$ then $\nabla u = 0$, so that u , and therefore φ , are necessarily constant. It follows from (3.3) and (3.4) that $0 \geq a(x)u \geq b(x)u^{\sigma+1} \geq 0$, so that, if a does not vanish identically, then $u \equiv 0$, without any assumption on b . On the other hand, if $A = H(K+1) - 1$, then the conclusion depends on the fact that b is positive somewhere.

Remark 3.3. We also note that if u is assumed to be strictly positive, then the conclusion of the Theorem holds, with a much easier proof, if we assume that $\max\{-1, A\} \leq H(K+1) - 1$, and that $\beta > -1$, $\beta \geq A$, $\beta \leq H(K+1) - 1$.

Remark 3.4. Let L_H be the Schrödinger operator defined by $L_H = \Delta + Ha(x)$. Then the validity of (3.3) is related to the sign of $\lambda_1^{L_H}(M)$. Indeed, assume that φ is a positive C^2 solution of (3.3). Let $\psi \in C_c^\infty(M)$ and apply the divergence theorem to the vector field $\psi^2 \nabla \log \varphi$. Since, by (3.3) and Young inequality

$$\begin{aligned} \operatorname{div}(\psi^2 \nabla \log \varphi) &= \frac{\psi^2}{\varphi} (\nabla \varphi - \frac{|\nabla \varphi|^2}{\varphi}) + 2 \frac{\psi}{\varphi} \langle \nabla \varphi, \nabla \psi \rangle \\ &\leq \frac{\psi^2}{\varphi} \left(-Ha(x)\varphi - (K+1) \frac{|\nabla \varphi|^2}{\varphi} \right) + \frac{\psi^2}{\varphi^2} |\nabla \varphi|^2 + |\nabla \psi|^2, \end{aligned}$$

we deduce that

$$\int_M |\nabla \psi|^2 - Ha(x)\psi^2 \geq K \int_M \psi^2 \frac{|\nabla \varphi|^2}{\varphi^2},$$

and from the variational characterization of the bottom of the spectrum we conclude that if $K \geq 0$ then $\lambda_1^{L_H}(M) \geq 0$.

On the other hand, if $\lambda_1^{L_H}(M) \geq 0$, then, by an extension of the result of Moss Pieperbrink, and Fisher-Colbrie Schoen quoted in Section 1 (see [PRS3], Lemma 1.2), there exists a positive C^1 function v which satisfies

$$\Delta v + Ha(x)v = 0$$

weakly on M . Further, if $a(x)$ is assumed to be $C^{0,\alpha}$ for some $\alpha \in (0,1)$, then v is C^2 and it is a classical solution of the above equation. It is clear that v is respectively a weak or a classical, solution of (3.3) for every $K \leq 0$.

Corollary 3.5. *Let $a(x), b(x) \in C^0(M)$ and assume that $b(x)$ is non-negative and does not vanish identically. Suppose also that, for some $H \geq 1$, $\lambda_1^{L_H}(M) \geq 0$. Then there are no positive C^2 solutions of the differential inequality*

$$(3.15) \quad \Delta u + a(x)u - b(x)u^\sigma \geq 0, \quad \sigma \in \mathbb{R}$$

such that

$$(3.16) \quad \left(\int_{B_r} u^{2(\beta+1)} \right)^{-1} \notin L^1(+\infty)$$

for some $0 \leq \beta \leq H-1$.

If $a(x)$ is assumed to be $C^{0,\alpha}$, the the corollary follows immediately from Remark 3.4 and from Theorem 3.1 with $K = 0$. In the general case, the function v satisfies inequality (3.3) with $K = 0$ only in weak sense, and the argument of Theorem 3.1 needs to be slightly modified to be carried out in this situation. Note also that, since the corollary deals with strictly positive solutions, we can drop the assumption that $\sigma \geq -1$.

In order to apply Theorem 3.1 and obtain the non-existence result mentioned at the beginning of this section, one needs to verify that the (non-)integrability condition (3.6) holds. In principle, this may be obtained combining a-priori upper estimates for u with appropriate bounds for the volume growth of balls. Both estimates can be deduced imposing lower bounds on the radial Ricci curvature of the manifold.

In the following lemma we deduce an a-priori integral estimate for nonnegative solutions of (3.4), which will enable us to obtain (3.6) under the sole assumption of a volume growth condition.

Lemma 3.6. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold, and let $a(x), b(x) \in C^0(M)$ with $b(x) > 0$ on M . Assume that $u \geq 0$ is a C^2 solution of the differential inequality*

$$(3.17) \quad u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2,$$

for $A \leq 1$ and $\sigma > 1$. Then for every $p \geq 1$, $p > A + 2$ there exist constants $C_1, C_2 > 0$ which depend only on p, σ and $R_0 > 0$ such that, for every $R \geq R_0$,

$$(3.18) \quad \int_{B_R} b(x)u^{p+\sigma-2} \leq C_1 R^{-2\frac{p+\sigma-2}{\sigma-1}} \int_{B_{2R}} b(x)^{-\frac{p-1}{\sigma-1}} + C_2 \int_{B_{2R}} \left(\frac{a_+(x)}{b(x)} \right)^{\frac{p-1}{\sigma-1}} a_+(x)$$

Proof. Observe first that we may assume that $u \not\equiv 0$, for otherwise there is nothing to prove. Thus, there exists $R_0 > 0$ such that $u \not\equiv 0$ on B_R for every $R \geq R_0$.

Next, for every $R \geq R_0$, let $\psi = \psi_R : M \rightarrow [0, 1]$ be a smooth cut-off function such that

$$(3.19) \quad \psi \equiv 1 \text{ on } B_R, \quad \psi \equiv 0 \text{ on } M \setminus B_{2R}, \text{ and } |\nabla \psi| \leq \frac{C}{R} \psi^{\frac{p-1}{p+\sigma-2}} \text{ on } B_{2R},$$

for some C which depends only on p and σ . Note that this is possible since the exponent $\frac{p-1}{p+\sigma-2}$ is strictly less than 1. Having fixed $\epsilon > 0$, we let W be the vector field defined by

$$W = \psi^2(u + \epsilon)^{p-3} u \nabla u.$$

A computation that uses (3.17) yields

$$\begin{aligned} \operatorname{div} W &\geq \psi^2(u + \epsilon)^{p-3} \left\{ -a(x)u^2 + b(x)u^{\sigma+1} + \left(1 - A - (p-3)\frac{u}{u+\epsilon}\right) |\nabla u|^2 \right\} \\ &\quad + 2\psi(u + \epsilon)^{p-3} u \langle \nabla u, \nabla \psi \rangle. \end{aligned}$$

We estimate the last term on the right hand side using Cauchy-Schwarz's inequality and Young's inequality $2ab \leq \lambda a^2 + \lambda^{-1}b^2$ with $\lambda = p-2-A > 0$, to obtain

$$\operatorname{div} W \geq \psi^2(u + \epsilon)^{p-3} \left\{ -a_+(x)u^2 + b(x)u^{\sigma+1} \right\} - \frac{1}{p-2-A} u(u + \epsilon)^{p-2} |\nabla \psi|^2.$$

We integrate the above inequality, apply the divergence theorem, rearrange, let $\epsilon \rightarrow 0+$ and use the dominated convergence theorem, in this order, to deduce that

$$(3.20) \quad \int_{B_{2R}} b(x)\psi^2 u^{p+\sigma-2} \leq \frac{1}{p-2-A} \int_{B_{2R}} u^{p-1} |\nabla \psi|^2 + \int_{B_{2R}} \psi^2 a_+(x) u^{p-1}$$

If $p = 1$ the conclusion follows immediately using (3.19). If $p > 1$, we denote by I and II the two integrals on the right hand side, and use Hölder inequality with conjugate exponents

$$\frac{p+\sigma-2}{p-1} (> 1) \quad \text{and} \quad \frac{p+\sigma-2}{\sigma-1},$$

and the assumption that $b(x) > 0$ to estimate

$$I \leq \left(\int_{B_{2R}} b(x)\psi^2 u^{p+\sigma-2} \right)^{\frac{p-1}{p+\sigma-2}} \left(\int_{B_{2R}} \psi^{-2\frac{p-1}{\sigma-1}} b(x)^{-\frac{p-1}{\sigma-1}} |\nabla \psi|^2 \frac{p+\sigma-2}{\sigma-1} \right)^{\frac{\sigma-1}{p+\sigma-2}}$$

and

$$II \leq \left(\int_{B_{2R}} b(x) \psi^2 u^{p+\sigma-2} \right)^{\frac{p-1}{p+\sigma-2}} \left(\int_{B_{2R}} \psi^2 a_+(x)^{\frac{p+\sigma-2}{\sigma-1}} b(x)^{-\frac{p-1}{\sigma-1}} \right)^{\frac{\sigma-1}{p+\sigma-2}}$$

Inserting into (3.20), noting that the integral on the left hand side is strictly positive by the choice of R , and simplifying, we obtain

$$\begin{aligned} \int_{B_{2R}} b(x) \psi^2 u^{p+\sigma-2} \leq & \left\{ \frac{1}{p-2-A} \left(\int_{B_{2R}} \psi^{-2\frac{p-1}{\sigma-1}} b(x)^{-\frac{p-1}{\sigma-1}} |\nabla \psi|^2 \right)^{\frac{\sigma-1}{p+\sigma-2}} \right. \\ & \left. + \left(\int_{B_{2R}} \psi^2 a_+(x)^{\frac{p+\sigma-2}{\sigma-1}} b(x)^{-\frac{p-1}{\sigma-1}} \right)^{\frac{\sigma-1}{p+\sigma-2}} \right\}^{\frac{p+\sigma-2}{\sigma-1}}. \end{aligned}$$

The required conclusion follows again using (3.19) and the elementary inequality $(a+b)^\tau \leq 2^\tau (a^\tau + b^\tau)$ valid for $a, b, \tau \geq 0$. \square

We are now ready for our main non-existence result.

Theorem 3.7. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold, and let $a(x), b(x) \in C^0(M)$ where $b(x) > 0$ on M and*

$$(3.21) \quad b(x) \geq \frac{C}{r(x)^\mu}$$

for $r(x) \gg 1$ and for some constants $C > 0$ and $0 \leq \mu \leq 2$. Assume that

$$(3.22) \quad \text{(i) } \sup_M \frac{a_+(x)}{b(x)} < +\infty \text{ and (ii) } \int_{B_r} a_+(x) = O(r^{2-\mu} \log r) \text{ as } r \rightarrow +\infty,$$

and that, for some $H \geq 1$, the operator $L_H = \Delta + Ha(x)$ satisfies

$$(3.23) \quad \lambda_1^{L_H}(M) \geq 0.$$

Finally, let A and σ be such that $A \leq 1$, $A < H-1$, $1 < \sigma \leq 2H+1$ and $\sigma < 2H-A$ and assume that

$$(3.24) \quad \text{vol } B_r = O\left(r^{2+(2-\mu)\frac{2H}{\sigma-1}} \log r\right) \text{ as } r \rightarrow +\infty.$$

Then the only non-negative C^2 solution u of the differential inequality

$$(3.25) \quad u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2$$

is $u \equiv 0$.

Proof. If we set $p = 2H + 2 - \sigma$, the conditions imposed on the parameters imply that p satisfies the assumptions listed in the statement of Lemma 3.6. The lemma and condition (3.22) (i) show that there exist constants $C_i > 0$ such that

$$(3.26) \quad \int_{B_r} b(x) u^{2H} \leq C_1 r^{-\frac{4H}{\sigma-1}} \int_{B_{2r}} b(x)^{1-\frac{2H}{\sigma-1}} + C_2 \int_{B_{2r}} a_+(x)$$

for $r > 0$ sufficiently large. We use condition (3.21) to estimate from below the integral on the left hand side. On the other hand, since $\sigma < 2H+1$, we may again use condition (3.21) to estimate from above the first integral on the right hand side, and (3.22) (ii) to estimate from above the second integral, and deduce that, for r sufficiently large,

$$\int_{B_r} u^{2H} \leq C(r^{(\mu-2)\frac{2H}{\sigma-1}} \text{vol } B_{2r} + r^2 \log r),$$

whence, using the volume growth condition (3.26) we conclude that

$$\int_{B_r} u^{2H} \leq Cr^2 \log r \quad \text{for } r \gg 1.$$

This immediately implies that, for r large,

$$\frac{r}{\int_{B_r} u^{2H}} \geq C \frac{1}{r \log r} \notin L^1(+\infty),$$

which in turn yields (see, e.g., [RS] Proposition 1.3)

$$\frac{1}{\int_{\partial B_r} u^{2H}} \notin L^1(+\infty).$$

We may therefore apply Theorem 3.1 with $K = 0$ and $\beta = H - 1$ to deduce that $\text{supp } u = \emptyset$, that is, $u \equiv 0$. \square

Remark 3.8. The argument used in the proof shows that the condition that $\frac{a_+}{b}$ is bounded above may be removed provided we replace (3.26) with

$$(3.27) \quad \int_{B_r} a_+^{\frac{2H}{\sigma-1}} = O\left(r^{2-\mu\frac{2H}{\sigma-1}} \log r\right) \quad \text{as } r \rightarrow +\infty.$$

Note that since the integral on the left hand side is a non-decreasing function of r this also imposes the further restriction $\mu \leq (\sigma - 1)/H$, with corresponding restrictions being imposed on the range of the other parameters.

Remark 3.9. In the case where the ambient manifold is Euclidean space, we can compare our Theorems 3.1 and 3.7 with the results in [AB], Section 3. We consider the equation

$$(3.28) \quad \Delta u + \lambda a(x)u - u^2 = 0 \quad \text{on } \mathbb{R}^m,$$

which, with $a(x) = g(x)$ and a change of scaling, is easily seen to be equivalent to (0.3). We assume, as in [AB], that $a(x)$ is positive somewhere, and that its positive part $a_+(x)$ satisfies the estimate

$$a(x) \leq \frac{k}{|x|^2},$$

for some positive constant k . According to the discussion at the end of Section 1, it follows that the principal eigenvalue λ_* of the linear equation associated to (3.28) is strictly positive and satisfies

$$\lambda_* \geq \frac{(m-2)^2}{4k}.$$

Moreover, if $\lambda \leq \lambda_*$, we have $\lambda_1^{\Delta + H\lambda a(x)}(\mathbb{R}^m) \geq 0$ provided $H \leq \frac{\lambda_*}{\lambda}$. On the other hand, if u is a non-negative solution of (3.28), Lemma 3.6 with $A = 0$, $\sigma = 2$ and $p > 2$ shows that

$$\int_{B_r} u^p \leq C \begin{cases} r^{m-2p} & \text{if } m-2p > 0 \\ \log r & \text{if } m-2p = 0 \\ 1 & \text{if } m-2p < 0, \end{cases}$$

and therefore

$$(3.29) \quad \frac{r}{\int_{B_r} u^p} \notin L^1(+\infty)$$

provided

$$p \geq \frac{m-2}{2}.$$

In order to apply Corollary 3.5, the non-integrability condition must hold with p satisfying

$$p = 2(\beta + 1) \leq 2H.$$

Summing up, if $\frac{\lambda_*}{\lambda} \geq \min\{1, \frac{m-2}{4}\}$ then Corollary 3.5 applies, and we conclude that every non-negative solution of (3.28) vanishes identically. To compare with Theorem 3.9 in [AB], we point out we are assuming a less stringent condition on a_+ and that we do not require that u tends to zero at infinity.

On the other hand, assume that $a_+(x)$ satisfies the more stringent condition assumed in [AB],

$$a_+(x) \leq \min\left\{\frac{k}{|x|^2}, \frac{A}{|x|^{2+\delta}}\right\}$$

for some positive constants A , k and δ . According to [AB], Theorem 3.5, every positive solution of (3.28) which tends to zero at infinity satisfies the estimate

$$(3.30) \quad u(x) \leq \frac{C}{|x|^{m-2}},$$

and, in fact, by Theorem 3.4 therein, every positive solution tends to zero at infinity, provided $a(x)$ is strictly negative off a compact. Now, it is easy to see that if u satisfies (3.30), then

$$\int_{B_r} u^2 \leq Cr^2,$$

and, clearly, $\lambda_1^{\Delta+\lambda a(x)}(\mathbb{R}^m) \geq 0$ for every $\lambda \leq \lambda_*$. An application of Corollary 3.5 with $\beta + 1 = H = 1$ shows that u vanishes identically. We therefore recover the conclusion of Theorem 3.9 in [AB].

Theorem 3.1 does not cover the "endpoint" case where $K = -1$ in (3.3), which we are going to consider presently. We therefore assume that there exists a positive solution φ of

$$(3.31) \quad \Delta\varphi + Ha(x)\varphi \leq \frac{|\nabla\varphi|^2}{\varphi}, \quad H > 0.$$

If u is a C^2 solution of (3.4) with $\sigma \geq 0$, we define $v = \varphi^{-\gamma}u$, $\gamma \geq 0$. A computation that uses (3.31), (3.4) and Young inequality yields

$$\begin{aligned} v\Delta v &= (\gamma H - 1)a(x)\varphi^{-2\gamma}u^2 + b(x)\varphi^{-2\gamma}u^{\sigma+1} \\ &\quad + \gamma^2\varphi^{-2\gamma-2}u^2|\nabla\varphi|^2 - A\varphi^{-2\gamma}|\nabla u|^2 - 2\gamma\varphi^{-2\gamma-1}u\langle\nabla u, \nabla\varphi\rangle \\ &\geq (\gamma H - 1)a(x)\varphi^{-2\gamma}u^2 + b(x)\varphi^{-2\gamma}u^{\sigma+1} \\ &\quad + \gamma^2\left(1 - \frac{1}{\epsilon}\right)\varphi^{-2\gamma-2}u^2|\nabla\varphi|^2 - (A + \epsilon)\varphi^{-2\gamma}|\nabla u|^2 \end{aligned}$$

Choosing $\gamma = 1/H$ and $\epsilon = -A$, the right hand side reduces to

$$b(x)\varphi^{-2/H}u^{\sigma+1} + \frac{1}{H^2}\left(1 + \frac{1}{A}\right)\varphi^{-2/H-2}u^2|\nabla\varphi|^2,$$

and we easily deduce that, if $A \leq -1$, then the function v satisfies

$$(3.32) \quad \Delta v \geq b(x)\varphi^{(\sigma-1)/H}v^\sigma.$$

Using (3.32) we obtain the following version of Theorem 3.1.

Theorem 3.10. *Let $a(x), b(x) \in C^0(M)$, with $b(x) \geq 0$, and assume that φ is a positive C^2 solution of (3.31) satisfying*

$$(3.33) \quad \varphi(x) \geq Cr(x)^{1/\delta}$$

for $r(x) \gg 1$, and some constants $C > 0$ and $\delta > 0$. Then the differential inequality

$$u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2,$$

with $\sigma \geq 0$ and $A \leq -1$, has no non-negative C^2 solution satisfying

$$(3.34) \quad \text{supp } u \cap \{x : M : b(x) > 0\} \neq \emptyset$$

and

$$(3.35) \quad \frac{r^{\delta p}}{\int_{\partial B_r} u^p} \notin L^1(+\infty)$$

for some $p > 1$.

Proof. According to (3.32) above, the function $v = \varphi^{-1/H}u$ is sub-harmonic. Further, (3.33) and (3.35) imply that

$$\left(\int_{\partial B_r} v^p \right)^{-1} \notin L^1(+\infty).$$

An application of Theorem B in [RS] shows that v is constant. The conclusion now follows as in the proof of Theorem 3.1. \square

Note that, even in the case of Theorem 3.10, if $A < -1$, then the conclusion can be strengthened to assert that every non-negative solution of (3.10) vanishes identically, unless $a(x) = b(x) \equiv 0$.

In applying Theorem 3.10 it is of course crucial to being able to find positive solutions of (3.31) satisfying the asymptotic lower bound (3.33). By contrast, in order to apply Theorem 3.1 one needs a positive solution of (3.3), whose existence, in typical applications like the one exemplified by Theorem 3.7 above, is guaranteed by means of assumptions on the spectrum of a suitable operator.

Observe now that if v is a solution of the Poisson equation

$$\Delta v = a(x)$$

then the function $\varphi = e^{-v}$ is a positive solution of

$$\Delta \varphi + a(x)\varphi = \frac{|\nabla \varphi|^2}{\varphi},$$

and furthermore, an upper bound for v yields a lower bound for φ .

The Poisson equation on complete Riemannian manifolds has been extensively studied using heat kernel techniques to obtain bounds on the Green kernel. To illustrate an application of Theorem 3.10, we consider the elementary case where the positive part of $a(x)$ is integrable. Then we have the following lemma (see, e.g., the proof of Theorem 3.2 in [NST])

Lemma 3.11. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete, non-parabolic manifold, and let $\rho \in C^{0,\alpha}(M) \cap L^1(M)$ ($0 \leq \alpha < 1$) be a non-negative function. Then, there exists a solution $v \in C^2$ of the Poisson equation*

$$\Delta v = \rho$$

satisfying $v \leq 0$.

Proof. Let $G(x, y)$ be the Green kernel, i.e., the minimal positive fundamental solution of the Laplacian, which exists by the assumption that M is non-parabolic. The Green kernel is symmetric and, if $\psi \in C_c^\infty(M)$, then the function $u(x) = -\int_M G(x, y)\psi(y)$ is smooth and satisfies $\Delta u = \psi$.

We claim that if $\rho \in C^0(M) \cap L^1(M)$ then the function $v = -\int_M G(x, y)\rho(y)$ is well defined and locally bounded. Assuming the claim, for every $\psi \in C_c^\infty(M)$ we have

$$\begin{aligned} \int_M v \Delta \psi &= - \int_M \psi(x) \int_M \Delta G(x, y) \rho(y) \\ &= - \int_M \rho(y) \int_M G(x, y) \Delta \psi(x) = \int_M \rho(y) \psi(y), \end{aligned}$$

so that v satisfies the Poisson equation in distributional sense, and therefore, by standard elliptic regularity (see [A]), Theorem 3.55), it is a classical solution. Clearly, v is non-positive.

To prove the claim, fix $R > 0$ and for every $x \in B_R$ we write

$$(3.36) \quad \int_M G(x, y) \rho(y) = \int_{B_{2R}} G(x, y) \rho(y) + \int_{M \setminus B_{2R}} G(x, y) \rho(y)$$

Since $G(x, y)$ is locally integrable uniformly for $x \in B_R$, the first integral on the right hand side is bounded above by a constant independent of $x \in B_R$.

On the other hand, by the local Harnack inequality there exists a constant C independent of $x \in B_R$ and such that

$$G(x, y) \leq CG(o, y) \quad \text{for every } y \in M \setminus B_{2R}.$$

Moreover,

$$\sup_{M \setminus B_{2R}} G(o, y) < +\infty.$$

Indeed, let Ω_n be an exhaustion of M by open sets containing o and with smooth boundary and let G_n be the Green kernel of Ω_n and recall that, by the standard construction of the Green kernel $G(x, y)$, $G_n(x, y) \rightarrow G(x, y)$ locally uniformly in $M \setminus \{x\}$. Let $C > \sup_{\partial B_{2R}} G(o, y)$, then, for every sufficiently large n we have $C > G(o, y) \geq G_n(o, y)$ for $y \in \partial B_{2R}$ and clearly $C > G_n(o, y) = 0$ for $y \in \partial \Omega_n$. Thus, by the comparison principle, $C > G_n(o, y)$ in $\Omega \setminus B_{2R}$, whence, letting $n \rightarrow +\infty$, $G(o, y) \leq C$ for $y \in M \setminus B_{2R}$. It follows that there exists a constant C' independent of $x \in B_R$ and $y \in M \setminus B_{2R}$ such that

$$G(x, y) \leq C'.$$

Since ρ is integrable, this implies that the second integral on the right hand side of (3.36) is also bounded independently of $x \in B_R$, as required to complete the proof of the claim. \square

Corollary 3.12. *Let (M, \langle, \rangle) be a complete, non-parabolic manifold, let the functions $a(x) \in C^{0,\alpha}(M)$, and $b(x) \in C^0(M)$ satisfy $b(x) > 0$ and*

$$(3.37) \quad a_+(x) \in L^1(M),$$

and suppose that for some constants $\sigma > 1$, $A \in (-\infty, -1]$ and μ, p, q satisfying

$$q > \max\{1, 3 - \sigma\} \quad 0 \leq \mu \leq 2 \frac{\sigma - 1}{\sigma + q - 2}, \quad p > \frac{q + \sigma - 2}{\sigma - 1},$$

we have

$$(3.38) \quad \int_{B_r} a_+(x)^p = O\left(r^{[2(\sigma-1)-\mu(q+\sigma-2)]\frac{p-1}{q-1}}\right) \quad \text{as } r \rightarrow +\infty$$

$$(3.39) \quad \text{vol } B_r = O\left(r^{2+(2-\mu)\frac{\sigma+q-2}{\sigma-1}}\right) \quad \text{as } r \rightarrow +\infty$$

$$(3.40) \quad b(x) \geq \frac{C}{r(x)^\mu} \quad \text{for } r(x) \gg 1.$$

Then there are no non-negative, non-identically zero $C^2(M)$ solutions of the differential inequality

$$(3.41) \quad u\Delta u + a(x)u^2 \geq b(x)u^{\sigma+1} - A|\nabla u|^2 \quad \text{on } M.$$

Proof. Since a_+ is integrable, by Lemma 3.11 and the preceding discussion there exists a solution $\varphi \geq 1$ of

$$\Delta\varphi + a_+\varphi = \frac{|\nabla\varphi|^2}{\varphi},$$

and φ is a solution of the differential inequality (3.31).

Now, let u be a non-negative solution of (3.41). Noting that $q > 1$, and $A \leq -1$ imply $q > A + 2$, applying Lemma 3.6 and using the lower bound for $b(x)$ (3.40) imply that

$$(3.42) \quad \int_{B_r} u^{q+\sigma-2} \leq C_1 r^{(\mu-2)\frac{q+\sigma-2}{\sigma-1}} \text{vol } B_{2r} + C_2 r^\mu \frac{q+\sigma-2}{\sigma-1} \int_{B_{2r}} a_+(x) \frac{\sigma+q-2}{\sigma-1}.$$

We claim that (3.38) implies

$$(3.43) \quad \int_{B_{2r}} a_+(x) \frac{\sigma+q-1}{\sigma-1} = O\left(r^{2-\mu\frac{\sigma+q-2}{\sigma-1}}\right) \quad \text{as } r \rightarrow +\infty,$$

which, together with the volume growth assumption (3.39) yields

$$\int_{B_r} u^{q+\sigma-2} = O(r^2) \quad \text{as } r \rightarrow +\infty.$$

As in the proof of Theorem 3.7, it follows (see, e.g., [RS] Proposition 1.3) that u satisfies condition (3.35) with $\delta = 0$ and exponent $q + \sigma - 2$ which is greater than 1 by the conditions on q . Thus, Theorem 3.10 (with $a(x)$ replaced by $a_+(x)$) applies and u vanishes identically.

To conclude it remains to prove the claim. To this end, we set $p' = (q + \sigma - 2)/(\sigma - 1)$, and apply Hölder inequality with conjugate exponents $(p - 1)/(p - p')$ and $(p - 1)/(p' - 1)$ to estimate

$$\begin{aligned} \int_{B_r} a_+(x)^{p'} &= \int_{B_r} a_+(x)^{\frac{p-p'}{p-1} + p\frac{p'-1}{p-1}} \\ &\leq \left(\int_{B_{2r}} a_+(x) \right)^{\frac{p-p'}{p-1}} \left(\int_{B_{2r}} a_+(x)^p \right)^{\frac{p'-1}{p-1}} \\ &= O\left(r^{[2(\sigma-1)-\mu(q+\sigma-2)]\frac{p'-1}{q-1}}\right) = O\left(r^{2-\mu\frac{q+\sigma-2}{\sigma-1}}\right), \end{aligned}$$

as required. \square

Remark 3.13. Assume that $b(x)$ satisfies the condition stated in the corollary, with $\mu < \sigma - 1$, and that conditions (3.38) and (3.39) are replaced by

$$\begin{aligned} \int_{B_r} a_+(x)^{\frac{2}{\sigma-1}} &= O\left(r^{2[1-\frac{\mu}{\sigma-1}]}\right) \quad \text{as } r \rightarrow +\infty \\ \text{vol } B_r &= O\left(r^{2[1-\frac{\mu-2}{\sigma-1}]}\right) \quad \text{as } r \rightarrow +\infty. \end{aligned}$$

It follows from (3.42) above with $q + \sigma - 2 = 2$, that every non-negative solution of

$$(3.44) \quad \Delta u + a(x)u - b(x)u^\sigma = 0$$

satisfies

$$(3.45) \quad \int_{B_r} u^2 \leq Cr^2 \log r,$$

and the same estimate is clearly satisfied by the difference of two solutions. An application of Theorem 4.1 in [BRS2] shows that (3.44) has at most one positive solution. We remark in this respect that if we replace u^2 in (3.45) with u^p with $p > 2$, then the conclusion of Theorem 4.1 in [BRS2] fails, as the example described on pages 214-215 therein shows.

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